Probabilistic Aspects of Fatigue

Introduction

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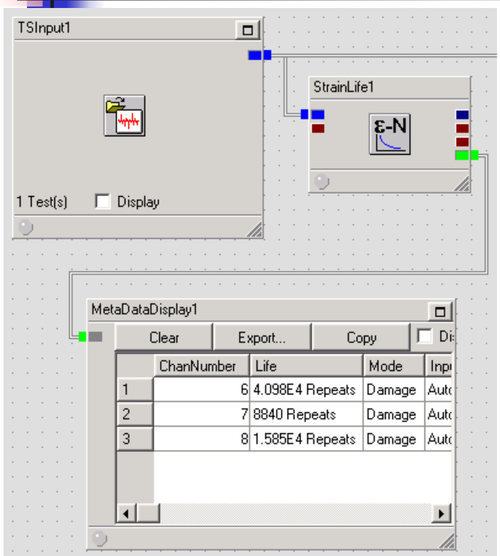
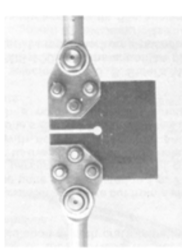
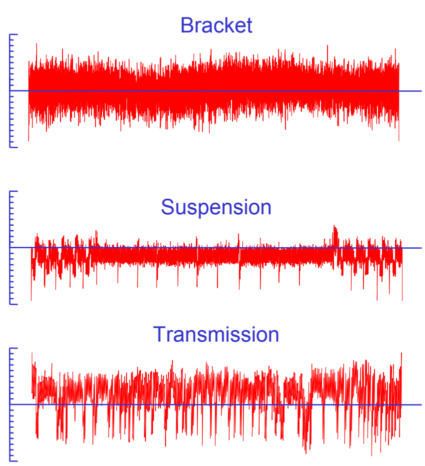
Fatigue Calculations



Who really believes these numbers ?

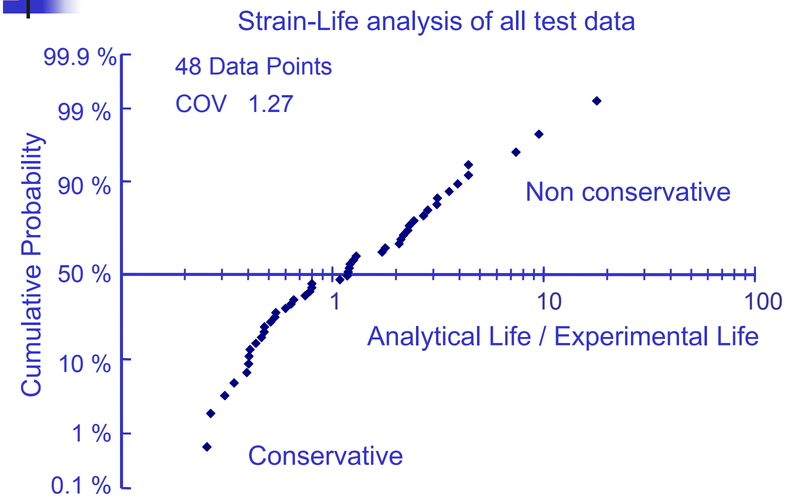
SAE Specimen



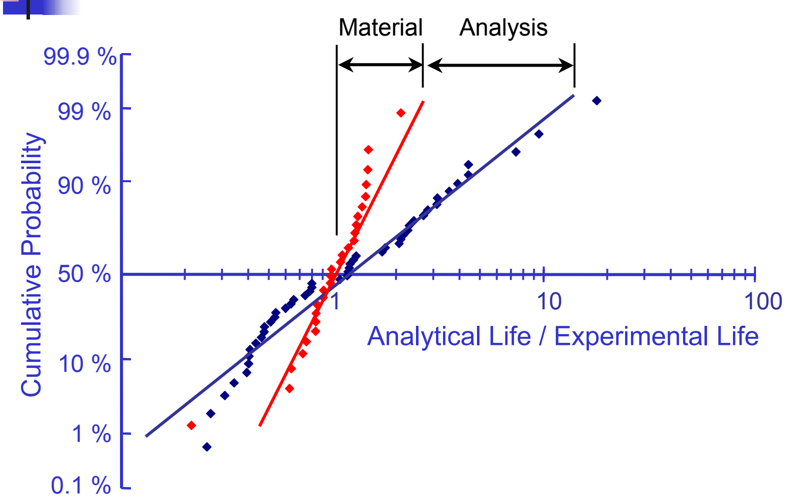


Fatigue Under Complex Loading: Analysis and Experiments, SAE AE6, 1977

Analysis Results



Material Variability



Strain-Life back calculation of specimen lives

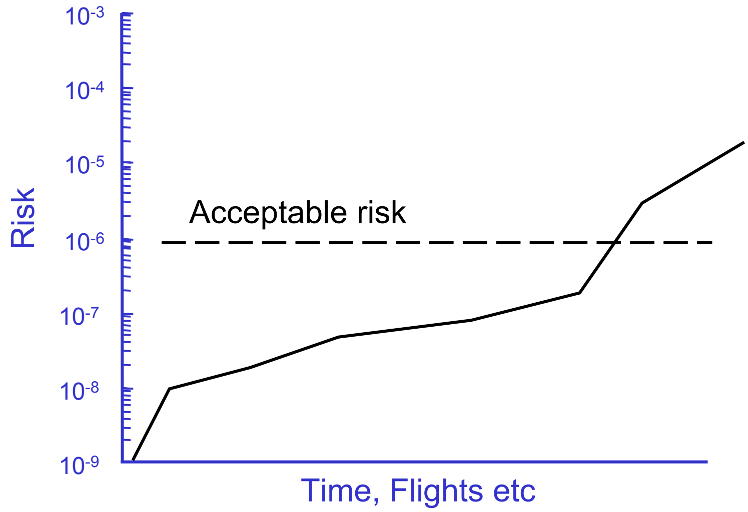
Probabilistic Models

- Probabilistic models are no better than the underlying deterministic models
- They require more work to implement
- Why use them?

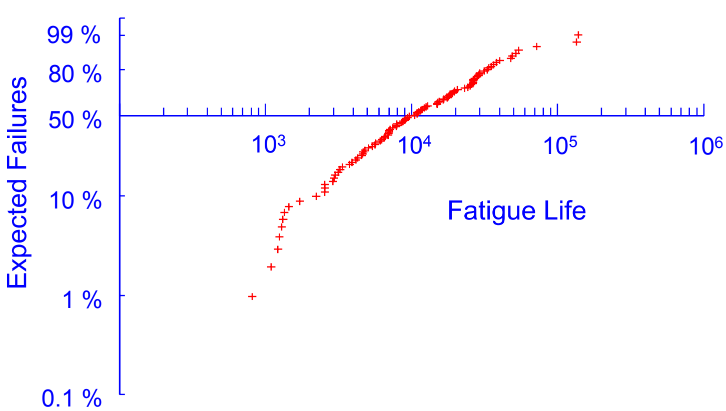
Quality and Cost

- Taguchi
  - Identify factors that influence performance
  - Robust design – reduce sensitivity to noise
  - Assess economic impact of variation
- Risk / Reliability
  - What is the increased risk from reduced testing ?

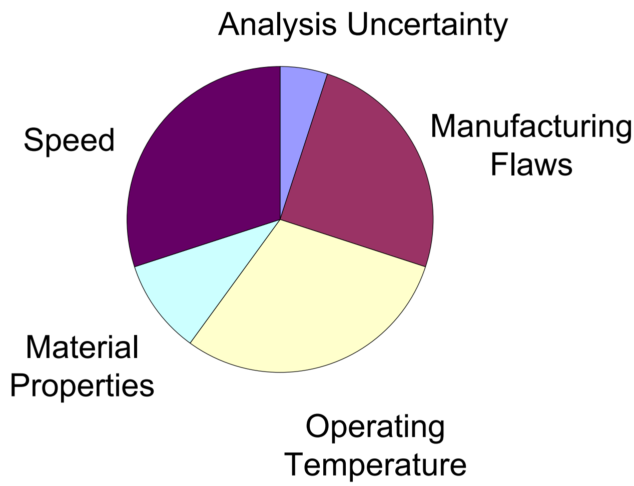
Risk



Reliability

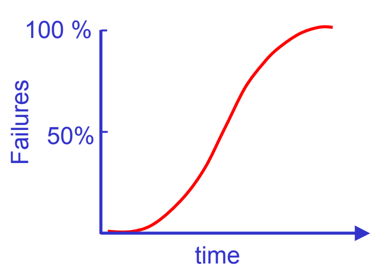


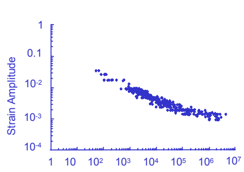
Risk Contribution Factors



Uncertainty and Variability

customers ← Stress → usage





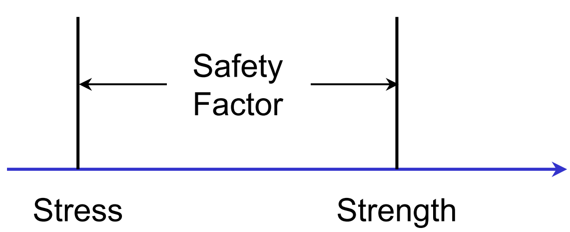
materials ← Strength → manufacturing

Deterministic versus Random

Deterministic – from past measurements the future position of a satellite can be predicted with reasonable accuracy

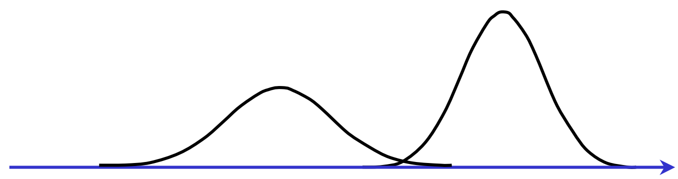
Random – from past measurements the future position of a car can only be described in terms of probability and statistical averages

Deterministic Design



Variability and uncertainty is accommodated by introducing safety factors. Larger safety factors are better, but how much better and at what cost?

Probabilistic Design



Reliability = 1 – P(Stress > Strength)

3σ Approach

3σ contains 99.87% of the data

$$P(s < S) = 2.3\ 10^{-3}$$

If we use 3σ on both stress and strength

$$P(\text{failure}) = P(\Sigma \geq s \ \cap \ s \leq S) = 5.3\ 10^{-6} \approx 4.5\sigma$$

The probability of the part with the lowest strength having the highest stress is very small

For 3 variables, each at 3 σ:

$$P(\text{failure}) = 1.2\ 10^{-8} \approx 5.7\sigma$$

Benefits

- Reduces conservatism (cost) compared to assuming the "worst case" for every design variable
- Quantifies life drivers – what are the most important variables and how well are they known or controlled ?
- Quantifies risk

Probabilistic Aspects of Fatigue

- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- Analysis Methods
- Characterizing Variability
- Case Studies
- FatigueCalculator.com
- GlyphWorks

Probabilistic Aspects of Fatigue

Basic Probability and Statistics

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Deterministic versus Random

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Random Variables

- Discrete - fixed number of outcomes
  - Colors
- Continuous - may have any value in the sample space
  - Strength

Descriptive Statistics

- Mean or Expected Value
- Variance / Standard Deviation
- Coefficient of Variation
- Skewness
- Kurtosis
- Correlation Coefficient

Mean or Expected Value

Central tendency of the data

$$\text{Mean} = \mu_x = \bar{x} = E(X) = \frac{\sum_{i=1}^{N} x_i}{N}$$

Variance / Standard Deviation

Dispersion of the data

$$\mathrm{Var}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}$$

Standard deviation

$$\sigma_x = \sqrt{\mathrm{Var}(X)}$$

Coefficient of Variation

$$COV = \frac{\sigma_x}{\mu_x}$$

Useful to compare different dispersions

μ = 10	μ = 100
σ = 1	σ = 10
COV = 0.1	COV = 0.1

Skewness

Skewness is a measure of the asymmetry of the data around the sample mean. If skewness is negative, the data are spread out more to the left of the mean than to the right. If skewness is positive, the data are spread out more to the right. The skewness of the normal distribution (or any perfectly symmetric distribution) is zero.

$$\text{Skewness}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^3}{N\sigma^3}$$

Kurtosis

Kurtosis is a measure of how outlier-prone a distribution is. The kurtosis of the normal distribution is 3. Distributions that are more outlier-prone than the normal distribution have kurtosis greater than 3; distributions that are less outlier-prone have kurtosis less than 3.

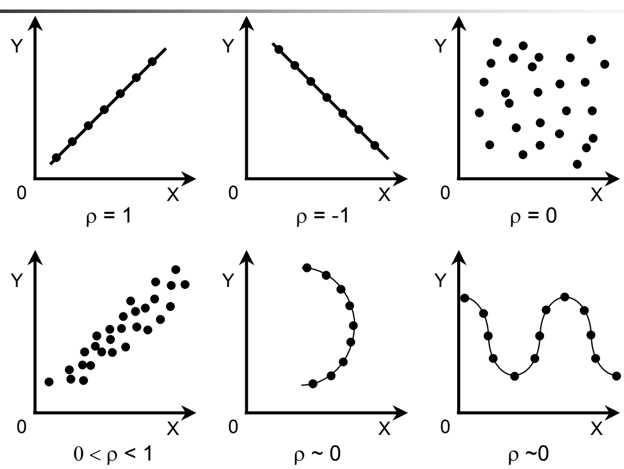
$$\text{Kurtosis}(X) = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^4}{N\sigma^4}$$

Covariance

A measure of the linear association between random variables

$$\sigma_{xy} = COV(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

Correlation Coefficient



$$\rho = \frac{E\left[(X-\mu_X)(Y-\mu_Y)\right]}{\sigma_X \sigma_Y}$$

Probability

- Basic probability
- Conditional probability
- Reliability

Basic Probability

The probability of event A occurring:

$$0 \le P(A) \le 1$$

$$P(A) = 1 \text{ certain}$$

$$P(A) = 0 \text{ impossssible}$$

Conditional Basic Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

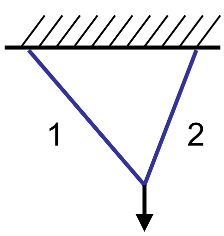
A or B — union

A and B — intersection

P(B) given A has occurred $$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independent Events



If A and B are unrelated

$$P(B|A) = P(B)$$

$$P(A|B) = P(A)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

Suppose the probability of bar 1 failing is 0.03
and the probability of bar 2 failing is 0.04.

What is the probability of the structure failing?

Failure Probability

Bar 1 or bar 2 fails

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A) = 0.03$$

$$P(B) = 0.04$$

$$P(A \cap B) = P(A) \cdot P(B) = 0.0012$$

P(failure) = 0.03 + 0.04 - 0.0012 = 0.0688

Reliability

P(no failure) = Reliability

Define $P(\bar{A})$ probability of not A

$$P(\bar{A}) = P(1 - A)$$

$$\text{Reliability} = P(\bar{A} \cap \bar{B})$$

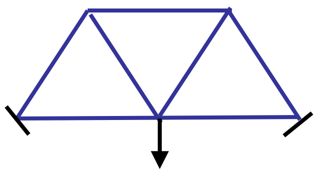
$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$$

For the 2 bar structure

$$P(\bar{A} \cap \bar{B}) = 0.97 \cdot 0.96 = 0.9312$$

$$P(\text{failure}) = 1 - \text{Reliability} = 0.0688$$

Structural Reliability



Collapse occurs if any member fails

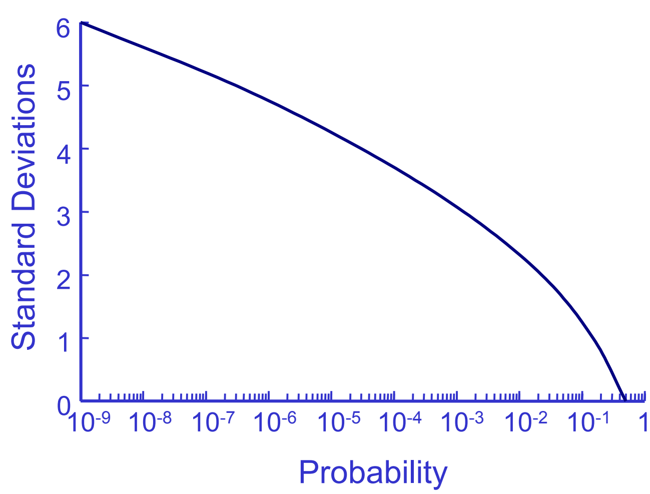
$$P(\overline{A}) = P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4 \cap \cdots \overline{A}_n)$$

$$P(\overline{A}) = P(\overline{A}_1) \cdot P(\overline{A}_2) \cdot P(\overline{A}_3) \cdot P(\overline{A}_4) \cdots P(\overline{A}_n)$$

Suppose $P(A) = 0.01$ for each link

$$P(\overline{A}) = (0.99)^7 = 0.932$$

Reliability



6 Sigma

6 sigma is 1 in a billion (0.999999999 Reliability)

Suppose a structure has 1000 bolted joints:

$$P(\bar{A}) = (0.999999999)^{1000} = 0.999999$$

1 in a million

3 sigma is (0.99865 Reliability)

$$P(\bar{A}) = (0.99865)^{1000} = 0.26$$

74 % failures

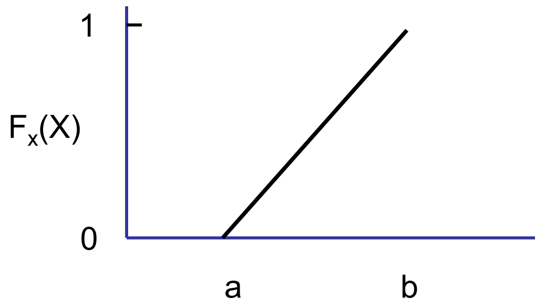
Statistical Distributions

- Uniform
- Normal
- LogNormal
- Gumble
- Weibull

A useful on-line reference:

http://www.itl.nist.gov/div898/handbook/

Cumulative Distribution Function

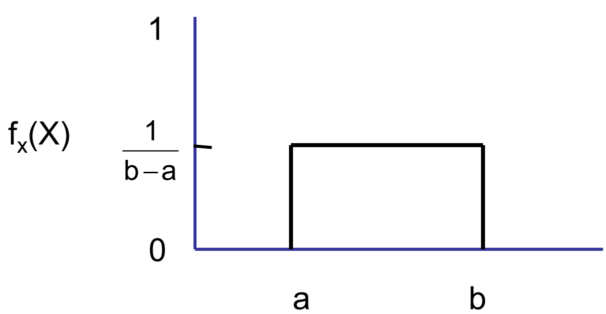


$$F_x(X) = P(X \le x)$$

$$F_x(-\infty) = 0$$

$$F_x(\infty) = 1$$

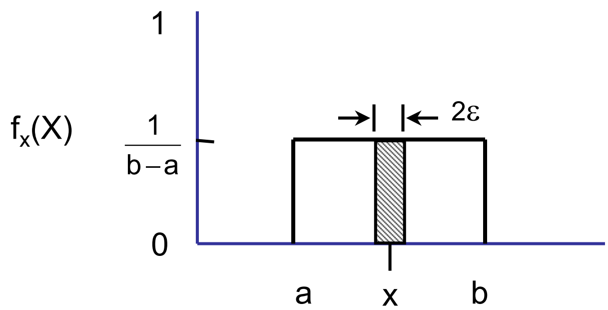
Probability Density Function



$$P(a < X \le b) = \int_a^b f_x(X)\,dx$$

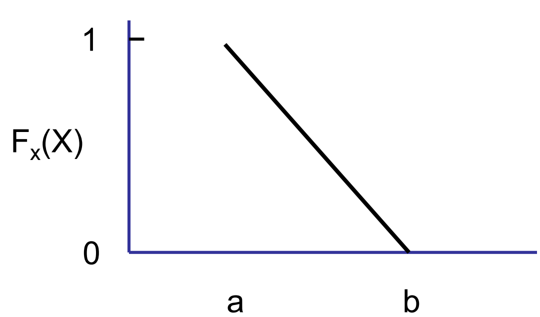
$$F_x(X) = \int_{-\infty}^{x} f_x(\xi)\,d\xi$$

Probability



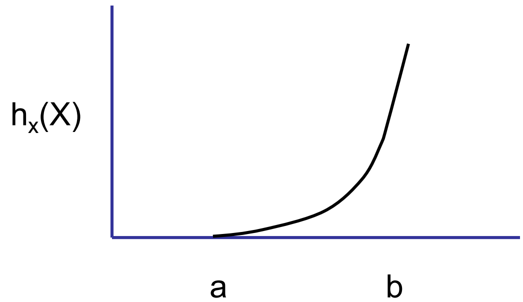
$$P\left(x-\varepsilon < X \le x+\varepsilon\right) = \int_{x-\varepsilon}^{x+\varepsilon} f_x(X)\,dx$$

Survival Function



$$S_x(X) = 1 - F_x(X)$$

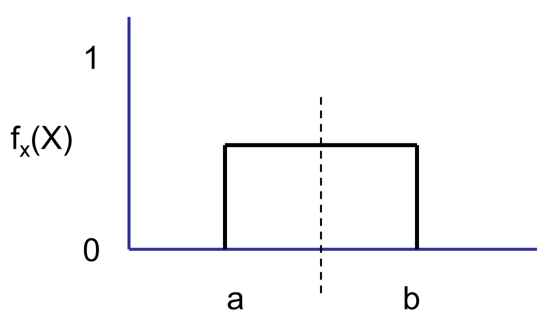
Hazard Function



$$h_x(X) = \frac{f_x(X)}{1 - F_x(X)}$$

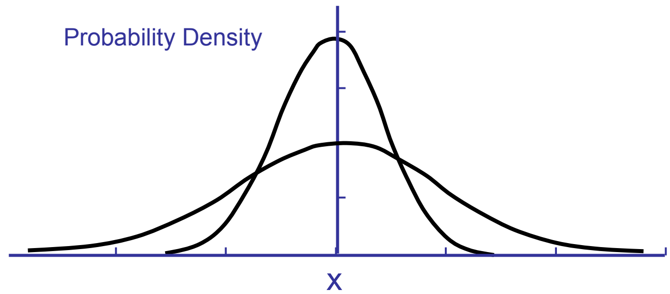
Instantaneous failure rate

Mean or Expected Value



$$\mu_x = E(X) = \int_{-\infty}^{\infty} x\, f_x(X)\, dx$$

Variance

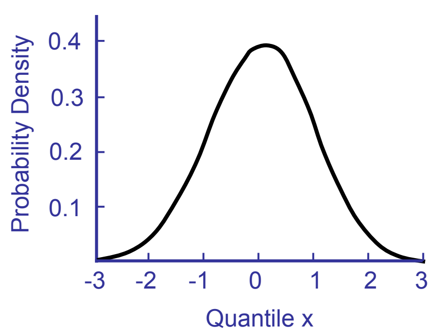


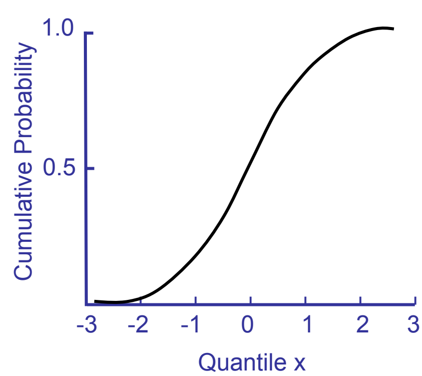
$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (X-\mu_x)^2 f_x(X)\,dx$$

$$\mathrm{Var}(X) = E\left(X^2\right) - \mu_x{}^2$$

$$\sigma_x = \sqrt{\mathrm{Var}(X)}$$

Normal Distribution

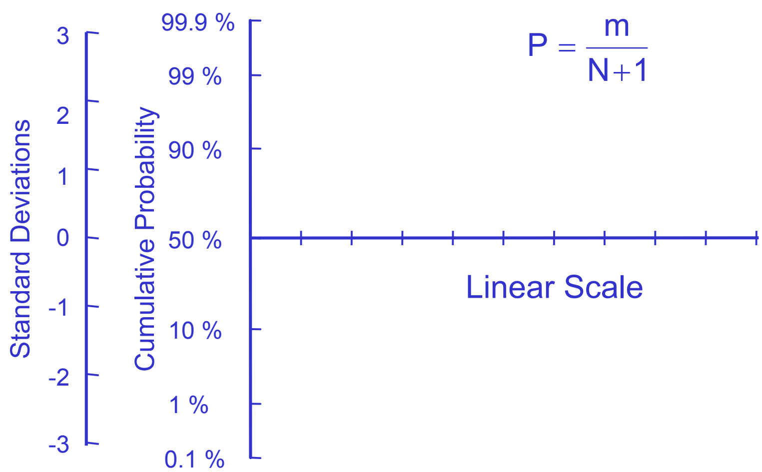




$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

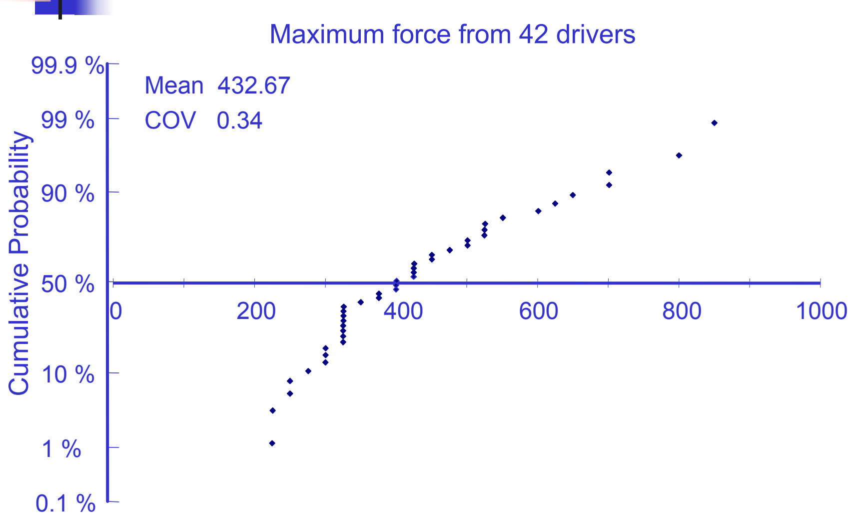
tables

Normal Probability Plot

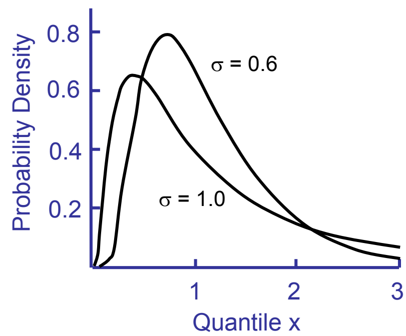


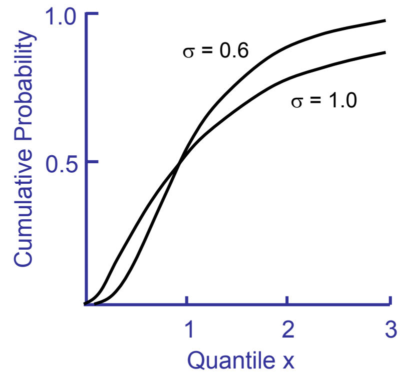
$$P = \frac{m}{N+1}$$

Normal Plot



LogNormal Distribution





$$m = \exp(\mu)$$

$$f_x(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[\frac{-\left(\ln\left(\frac{x}{m}\right)\right)^2}{2\sigma^2}\right]$$

tables

Useful Relationships

$$\mu_{\ln x} = \ln\left(\frac{\mu_x}{\sqrt{1+COV_x^{\,2}}}\right)$$

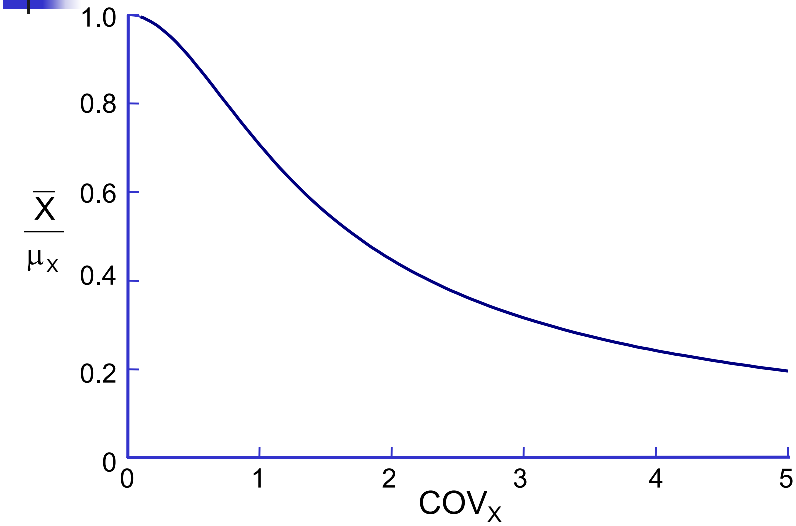
$$\sigma_{\ln x}^{\,2} = \ln\left(1+COV_x^{\,2}\right)$$

$$COV_x = \sqrt{\exp(\sigma_{\ln x}^{\,2})-1}$$

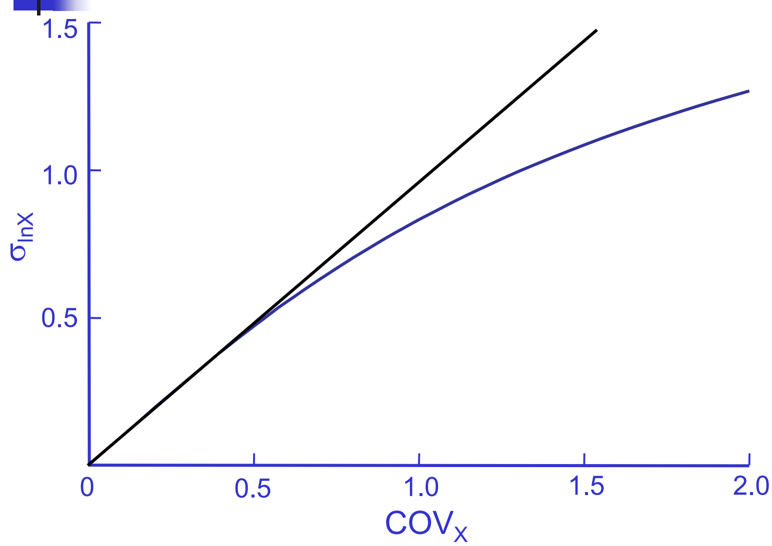
$$\mu_x = \exp\left(\mu_{\ln x} + 0.5\,\sigma_{\ln x}^{\,2}\right)$$

$$\bar{X}_x = \exp(\mu_{\ln x}) = \frac{\mu_x}{\sqrt{1+COV_x^{\,2}}}$$

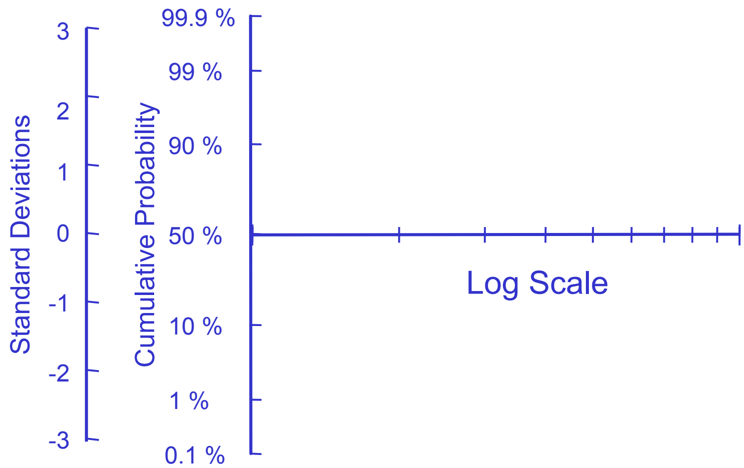
Median / Mean ratio



σ_{lnX} - COV_X

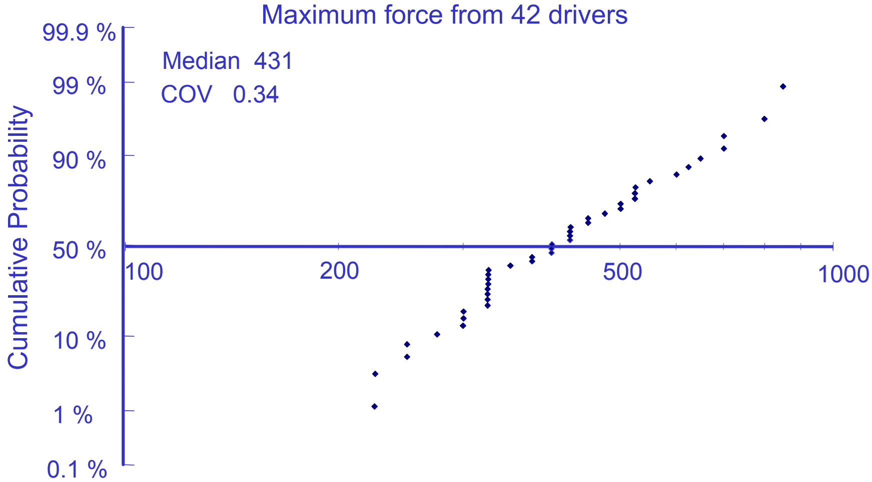


LogNormal Probability Plot

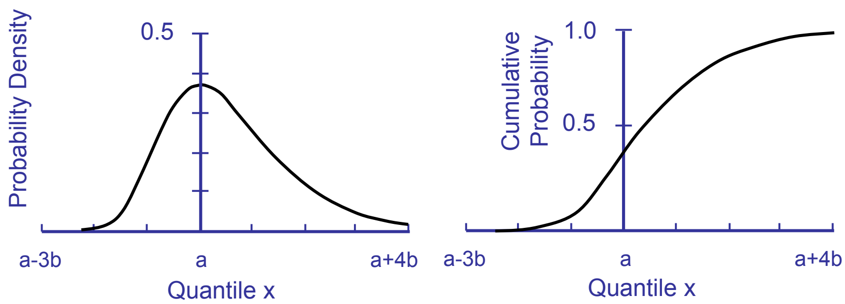


LogNormal Plot

Maximum force from 42 drivers



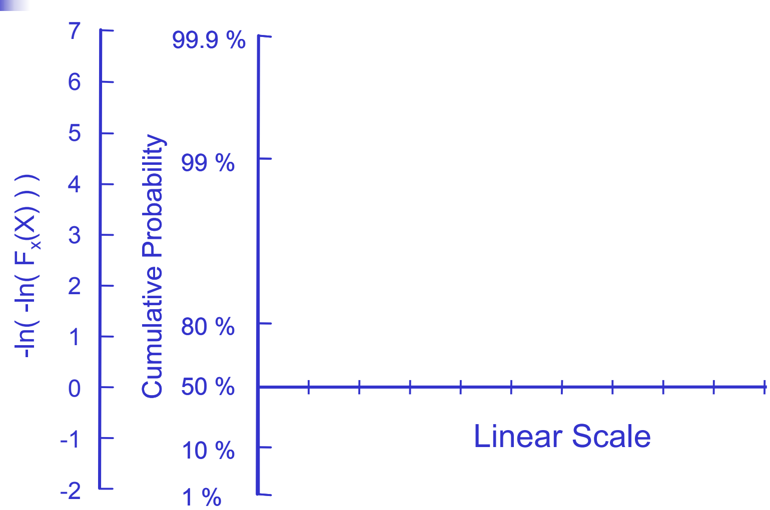
Gumbel Extreme Value



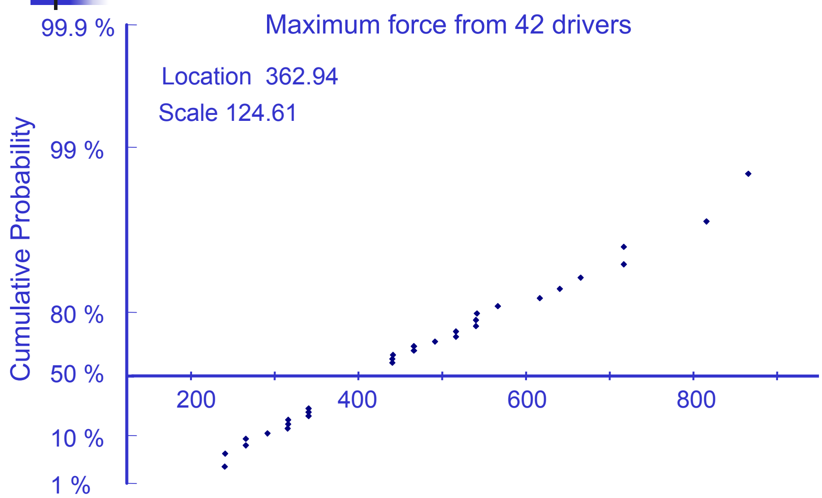
$$f_x(x) = \frac{1}{b}\exp\left[\frac{-(x-a)}{b}\right]\exp\left\{-\exp\left[\frac{-(x-a)}{b}\right]\right\}$$

$$F_x(x) = \exp\left\{-\exp\left[\frac{-(x-a)}{b}\right]\right\}$$

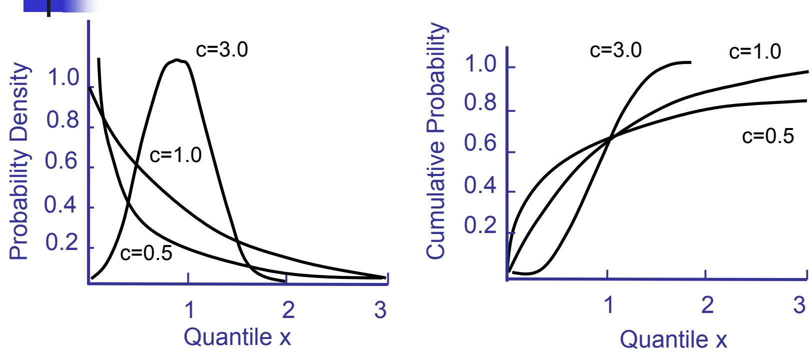
Gumble Probability Plot



Gumble Plot

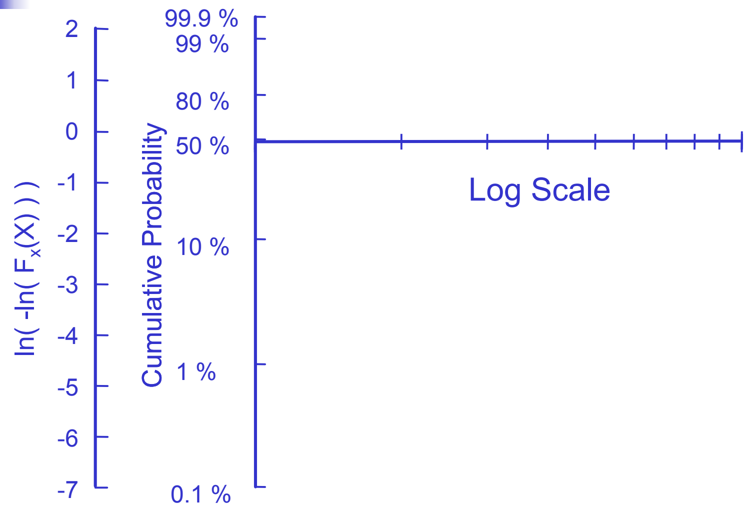


Weibull

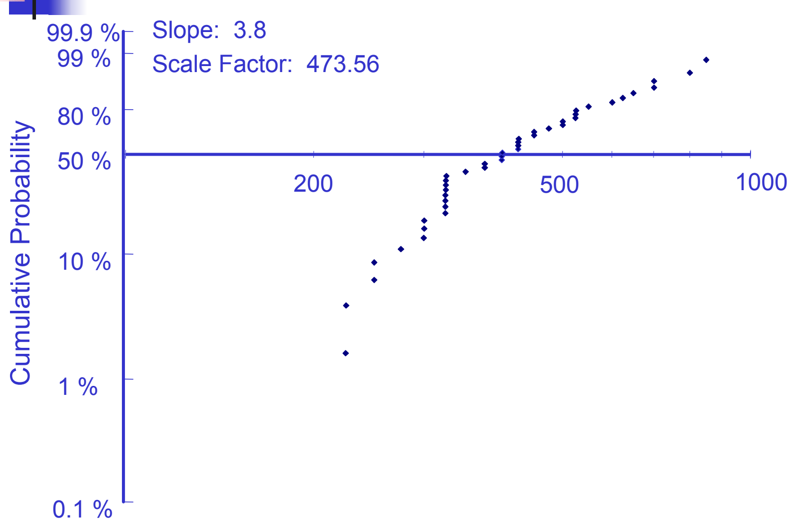


$$f_x(x) = \left(\frac{c\,x^{c-1}}{b^c}\right)\exp\left[-\left(\frac{x}{b}\right)^c\right] \qquad F_x(x) = 1 - \exp\left[-\left(\frac{x}{b}\right)^c\right]$$

Weibull Probability Plot



Weibull Plot

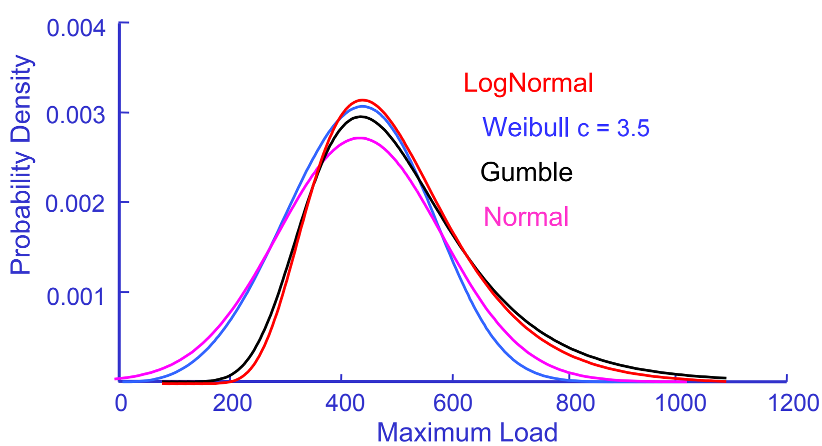


99.9 % Probability

Estimated maximum force from the distributions

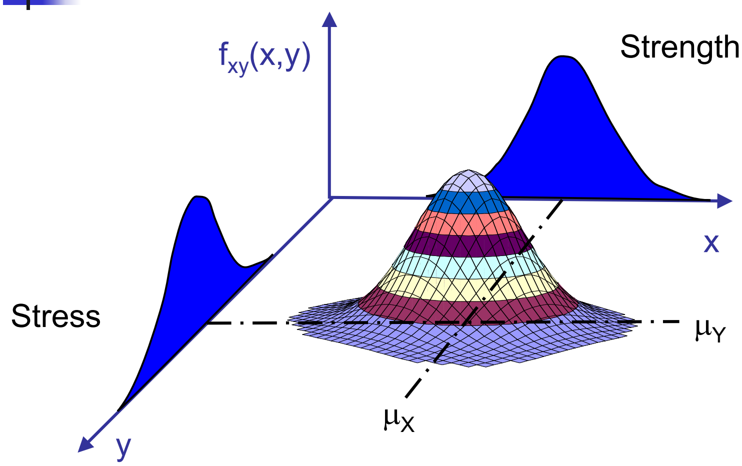
Normal	893
LogNormal	1125
Gumbel	1223
Weibull	785

Comparison



All distributions are approximately normal around the median

Joint Probability Density



Reliability = 1 – P(Stress > Strength)

Joint Probability Density

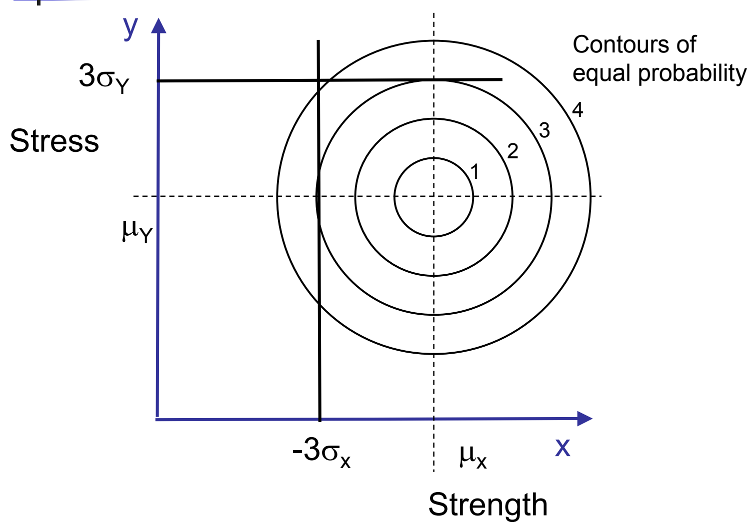
Normal distributions

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$$\exp\left[\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right]$$

ρ correlation coefficient

2D - Joint Probability Density



Acceptable Risk

- Safety
  - P(failure) ~ $10^{-6} \rightarrow 10^{-9}$
- Economic
  - P(failure) ~ $10^{-2} \rightarrow 10^{-4}$

Extreme Values

For most durability problems, we are not interested in the "large extremes" of stress or strength. Failure is much more likely to come from moderately high stresses combined with moderately low strengths.

Key Points

- Basic Nomenclature
  - Random Variables
    - Statistical Distribution
      - Cumulative distribution function
      - Mean
      - Variance
  - Probability
    - Marginal
    - Conditional
    - Joint

Probabilistic Aspects of Fatigue

Statistical Techniques

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Department of Mechanical and Industrial Engineering

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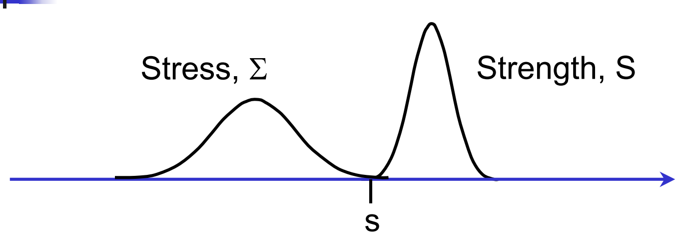
Statistical Techniques

- Normal Distributions
- LogNormal Distributions
- Monte Carlo
- Sampling
- Distribution Fitting

Failure Probability



Let Σ be the stress and S the fatigue strength

Given the distributions of Σ and S find the probability of failure

$$P(\ \Sigma \geq s \cap s \leq S\ )$$

Normal Variables

Linear Response Function

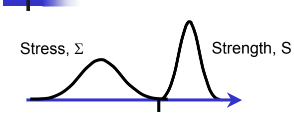
$$Z = a_o + \sum_{i=1}^{n} a_i X_i$$

$$X_i \sim N(\mu_i , C_i)$$

$$\mu_z = a_o + \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_i^2}$$

Calculations



$S \sim N(200 , 0.1)$ $\sigma_S = 20$

$\Sigma \sim N(100 , 0.2)$ $\sigma_\Sigma = 20$

Safety factor of 2

Let Z be a random variable:

$$Z = S - \Sigma$$

$$\mu_z = \mu_S - \mu_\Sigma$$

$$\mu_z = 200 - 100 = 100$$

$$\sigma_z = \sqrt{\sigma_S^2 + \sigma_\Sigma^2}$$

$$\sigma_z = \sqrt{20^2 + 20^2} = 28.2$$

Failure Probability

$$Z = S - \Sigma$$

Failure will occur whenever Z <= 0

$$Z = \mu_z - z\,\sigma_z = 0$$

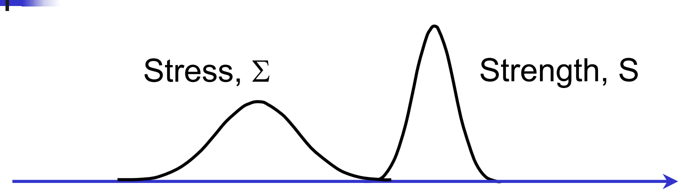
$$z = \frac{\mu_Z}{\sigma_Z} = \frac{100}{28.2}$$

z = 3.54 standard deviations

P(failure) = 2×10^{-4}

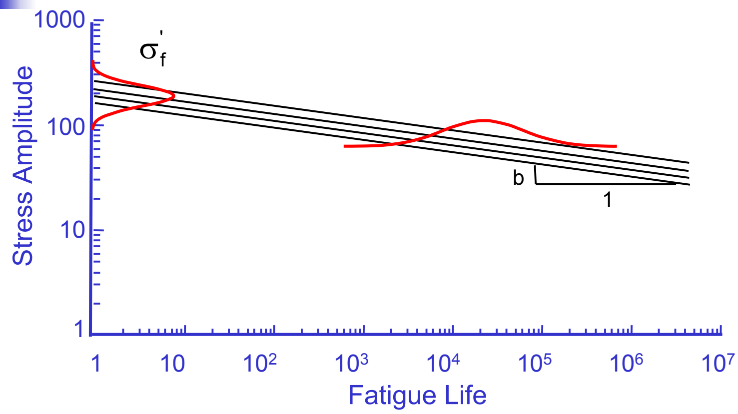
For this case only, a safety factor of 2 means a probability of failure of 2×10^{-4}. Other situations will require different safety factors to achieve the same reliability.

Failure Distribution



What is the expected distribution in fatigue lives?

Fatigue Data



$$\sigma_f' = \frac{\Delta S}{2(2N_f)^b}$$

$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

LogNormal Variables

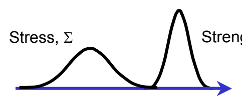
$$Z = a_o \prod_{i=1}^{n} X_i^{a_i}$$

a's are constant and $X_i \sim LN(\, x_i \,,\, C_i \,)$

median $\bar{Z} = a_o \prod_{i=1}^{n} \bar{X}_i^{a_i}$

COV $C_Z = \sqrt{\prod_{i=1}^{n} \left(1 + C_{X_i}^{\,2}\right)^{a_i^2} - 1}$

Calculations



$\sigma_f' \sim LN(\ 1000\ ,\ 0.1\ )\ \sigma = 100$

$\frac{\Delta S}{2} \sim LN(\ 250\ ,\ 0.2\ )\ \sigma = 50$

$b = -0.125$

$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

$$Z = 2N_f = \left(\frac{\Delta S}{2}\right)^{-8} \sigma_f'^{\,8}$$

$$\bar{Z} = 2N_f = \left(\frac{\overline{\Delta S}}{2}\right)^{-8} \bar{\sigma}_f'^{\,8}$$

$$COV_Z = \sqrt{\left(1+COV_{\Delta S}^{\ 2}\right)^{-8^2}\left(1+COV_{\sigma_f}^{\ 2}\right)^{8^2} - 1}$$

Results

	ΔS/2	σ_f'	**$2N_f$**		**Percentile**	**Life**
μ_x	250	1000	355,368		99.9	17,706,069
COV_x	0.2	0.1	4.72		99	4,566,613
					95	1,363,200
μ_{lnx}	5.50	6.90	11.21		90	715,589
X	245	995	73,676		50	73,676
σ_x	50	100	1,676,831		10	7,586
σ_{lnx}	0.198	0.100	1.774		5	3,982
					1	1,189
b =	-0.125				0.1	307

Monte Carlo Methods

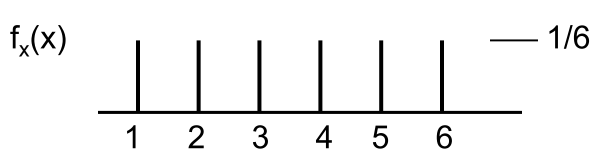
$$\frac{K_f \Delta S}{2} = \sqrt{E\left(\frac{\sigma_f'^{\,2}}{E}(2N_f)^{2b} + \sigma_f' \varepsilon_f' (2N_f)^{b+c}\right)}$$

Given random variables for K_f, ΔS, σ_f' and ε_f'
Find the distribution of $2N_f$

$Z = 2N_f = ?$

Simple Example

Probability of rolling a 3 on a die



Uniform discrete distribution

Computer Simulation

1. Generate random numbers between 1 and 6, all integers

2. Count the number of 3's

Let X_i = 1 if 3
0 otherwise

$$P(3) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

EXCEL

=ROUNDUP(6 * RAND() , 0)

=IF(A1 = 3 , 1 , 0)

=SUM(B1:B1)/ROW(B1)

5	0	0
3	1	0.5
4	0	0.333333
4	0	0.25
5	0	0.2
6	0	0.166667
1	0	0.142857
3	1	0.25
3	1	0.333333
6	0	0.3

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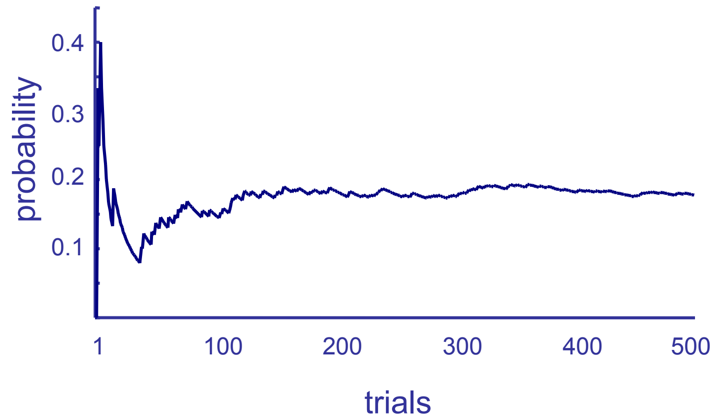
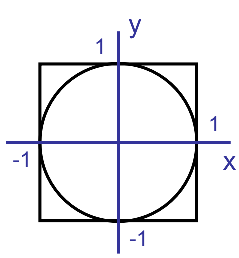
Results



Evaluate π



P(inside circle)

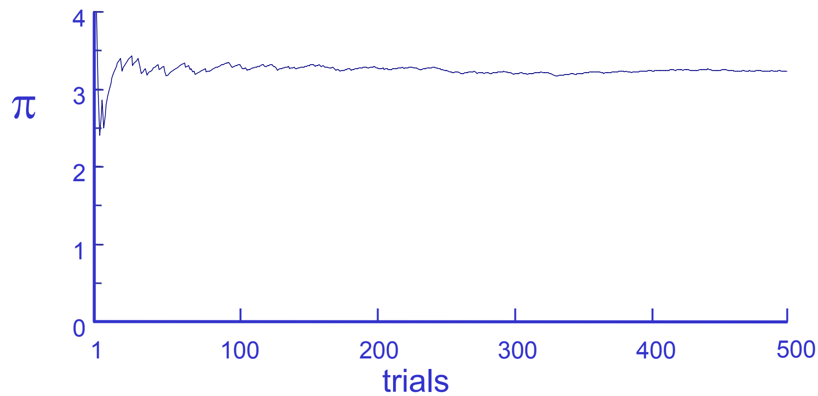
$$P = \frac{\pi r^2}{4}$$

$$\pi = 4\,P$$

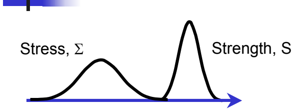
x = 2 * RAND() - 1
y = 2 * RAND() - 1

IF($x^2 + y^2 < 1$, 1 , 0)

π



Monte Carlo Simulation



$$2N_f = \left(\frac{\Delta S}{2\sigma_f'}\right)^{\frac{1}{b}}$$

$\sigma_f' \sim LN(\,1000\;,\;0.1\,)\;\sigma = 100$

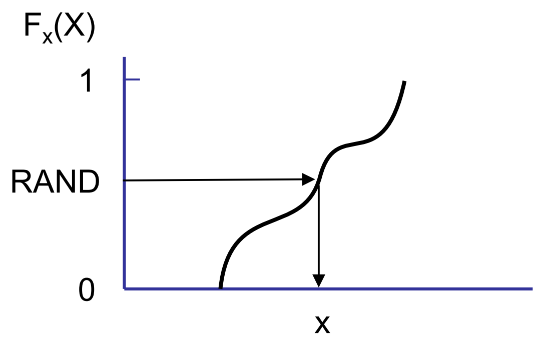
$\frac{\Delta S}{2} \sim LN(\,250\;,\;0.2\,)\;\sigma = 50$

$b = -0.125$

Randomly choose values of S and σ_f' from their distributions

Repeat many times

Generating Distributions



Randomly choose a value between 0 and 1

$$x = F_x^{-1}(\text{ RAND })$$

Generating Distributions in EXCEL

Normal

=NORMINV(RAND(),μ,σ)

Log Normal

=LOGINV(RAND(),lnμ,lnσ)

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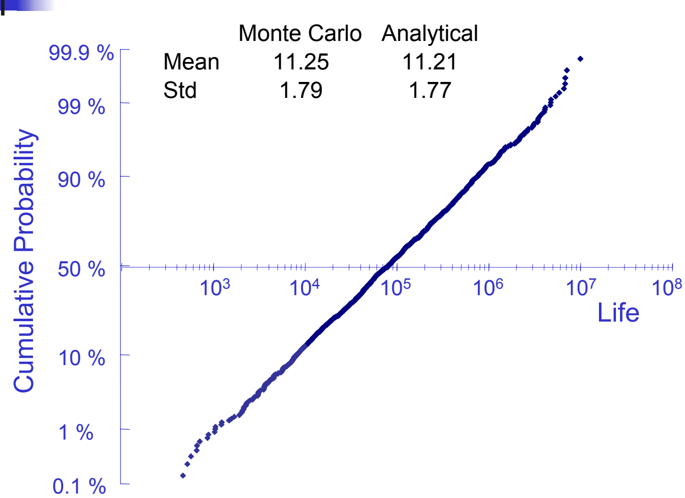
EXCEL

σ_f'	$\frac{\Delta S}{2}$	$2N_f$
893	204	134,677
1102	301	32,180
852	285	6,355
963	173	929,249
1050	283	35,565
1080	265	77,057
965	313	8,227
1073	213	420,456
1052	226	224,000
954	322	5,878
965	240	68,671
993	207	277,192
1191	368	11,967
831	210	59,473

Simulation Results



Summary

Simulation is relatively straightforward and simple

Obtaining the necessary input data is difficult

Nomenclature

Population: the totality of the observations

Sample: a subset of the population

Population mean: μ_x

Population variance: σ_x^2

Sample mean: $\overline{X}$

Sample variance: s^2

$$E(\overline{X}) = \mu_x$$

$$E(s^2) = \sigma_x^2$$

Sample Mean and Variance

Mean $$\overline{X} = \frac{\sum_{1}^{n} x}{n}$$

in EXCEL =AVERAGE(A1:A_n)

Variance $$s^2 = \frac{\sum_{1}^{n} (x - \overline{X})}{n-1}$$

in EXCEL =STDEV(A1:A_n)

Confidence Intervals

$$E(\overline{X}) = \mu_x$$

What is the probability that a sample $\overline{X}$ is greater than μ_x? 50%

$$P(L <= \mu_x <= U) = 1 - \alpha$$

There is a $1 - \alpha$ chance of selecting a sample in the interval between L and U that contains the true mean of the population

Translation

90% confidence

If we sampled a population many times to estimate the mean, 90% of the time the true population mean would lie between the computed upper and lower limit.

Confidence Interval - mean

For a normal distribution:

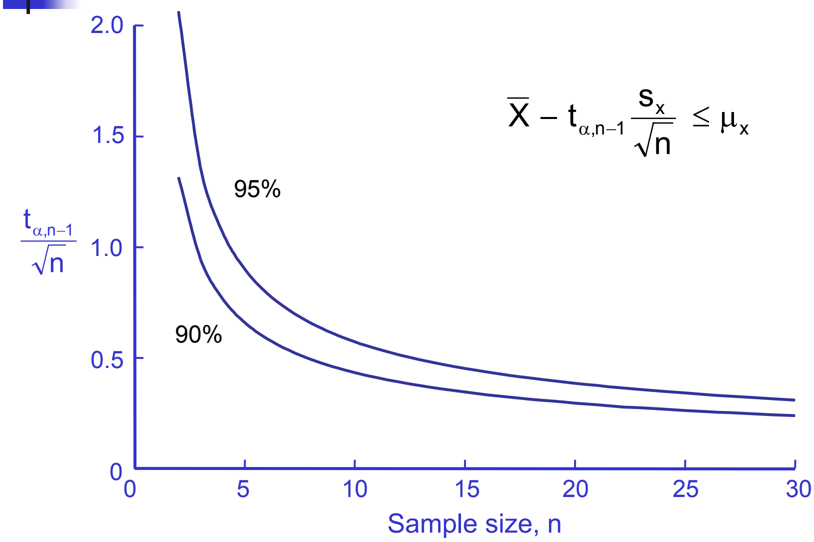
Lower limit of μ

$$\overline{X} - t_{\alpha,n-1}\frac{s_x}{\sqrt{n}} \leq \mu_x$$

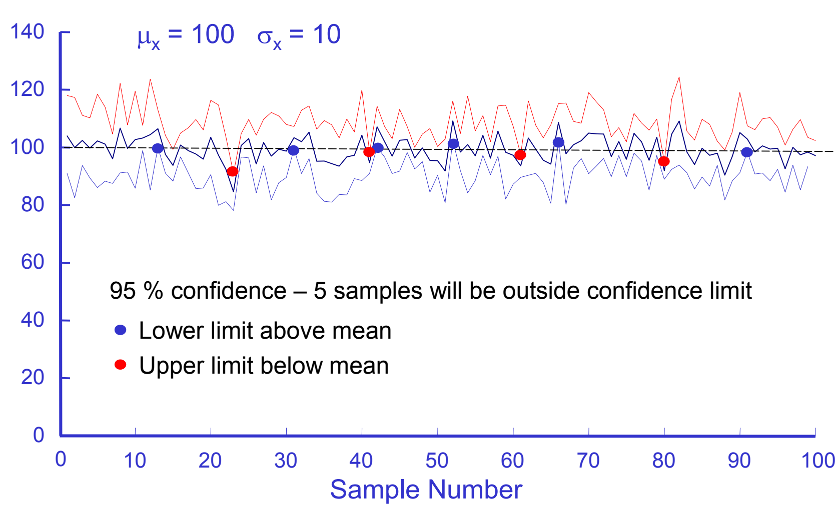
Upper limit of μ

$$\overline{X} + t_{\alpha,n-1}\frac{s_x}{\sqrt{n}} \geq \mu_x$$

Sample Size



5 Samples from Normal Distribution



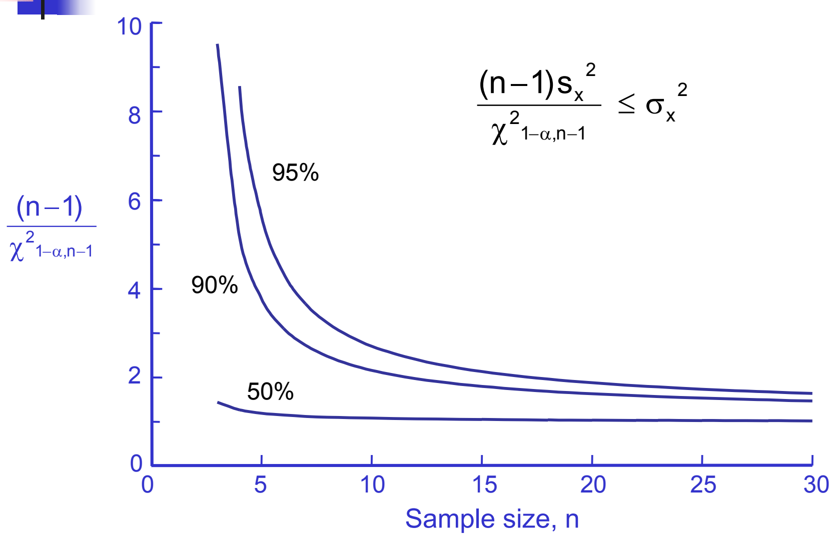
Confidence Interval - variance

For a normal distribution:

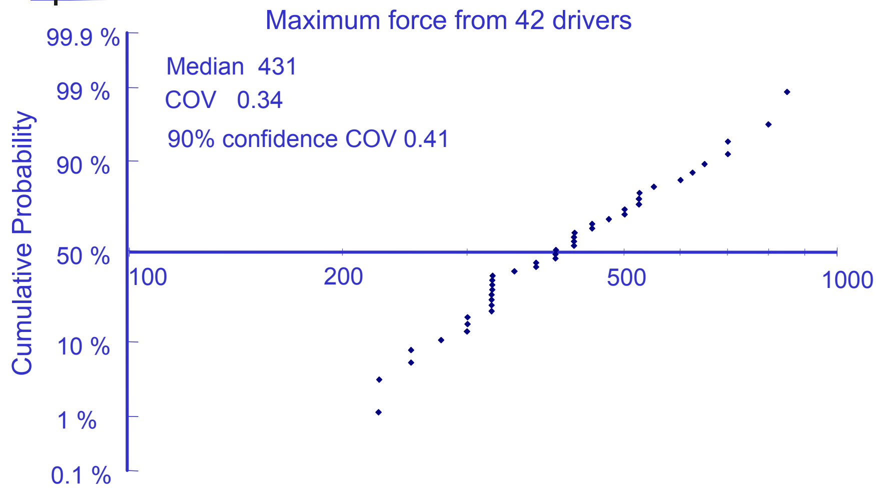
Upper limit of σ

$$\frac{(n-1)s_x^2}{\chi^2_{1-\alpha,n-1}} \le \sigma_x^2$$

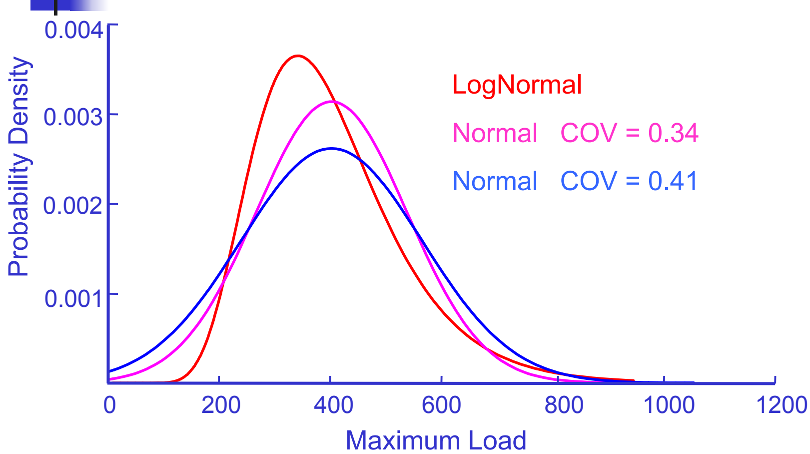
Sample Size



Maximum Load Data



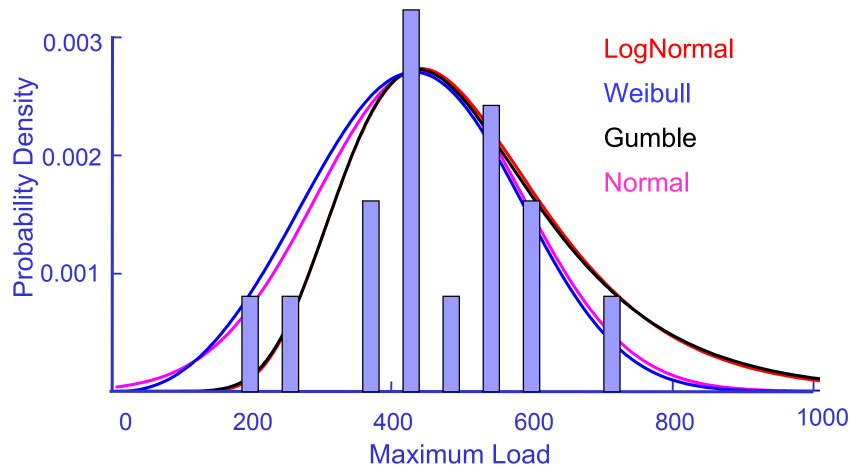
Maximum Load Data



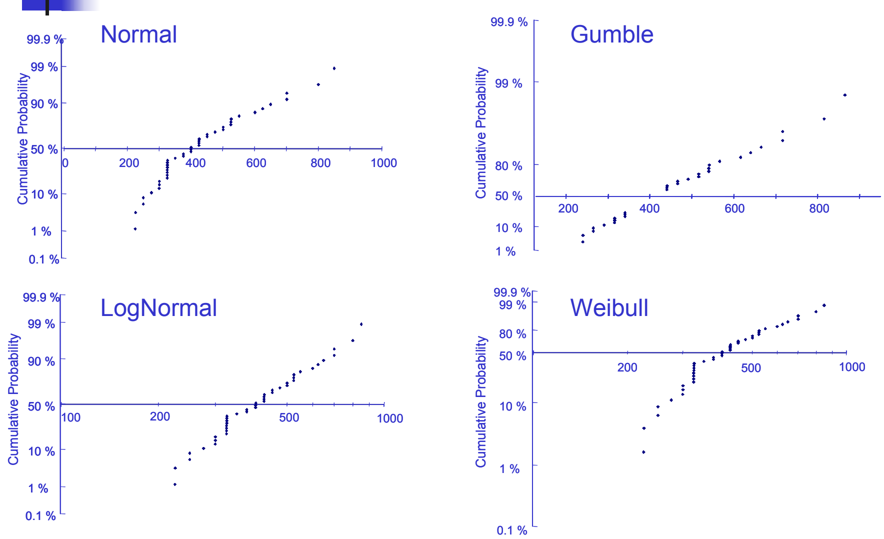
Uncertainty in Variance is just as important,
perhaps more important than the choice of the distribution

Choose the “Best” Distribution

15 samples from a Normal Distribution



Goodness of Fit Tests



Analytical Tests

Chi-square

any univariate distribution

Snedecor, George W. and Cochran, William G. (1989), *Statistical Methods, Eighth Edition*, Iowa State University Press.

Kolmogorov-Smirnov

tends to be more sensitive near the center of the distribution

Chakravarti, Laha, and Roy, (1967). *Handbook of Methods of Applied Statistics, Volume I*, John Wiley and Sons, pp. 392-394.

Anderson-Darling

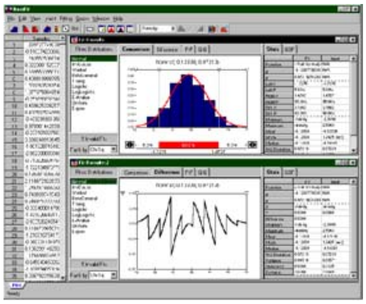
gives more weight to the tails

Stephens, M. A. (1974). *EDF Statistics for Goodness of Fit and Some Comparisons*, Journal of the American Statistical Association, Vol. 69, pp. 730-737.

Distributions

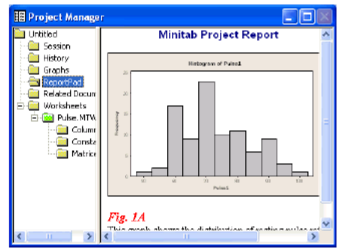
- Normal
  - Strength
  - Dimensions
- LogNormal
  - Fatigue Lives
  - Large variance in properties or loads
- Gumble
  - Maximums in a population
- Weibull
  - Fatigue Lives

Statistics Software



BestFit

www.palisade.com



Minitab

www.minitab.com

Central Limit Theorem

If X_1, X_2, X_3 ….. X_n is a random sample from the population, with sample mean $\overline{X}$, then the limiting form of

$$Z = \frac{\overline{X} - \mu_X}{\sigma / \sqrt{n}}$$

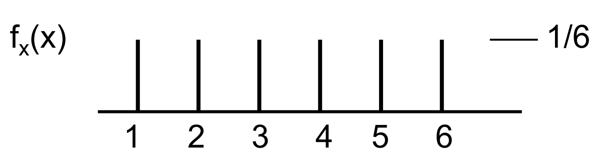
as $n \to \infty$ is the standard normal distribution

Translation

When there are many variables affecting the outcome, The final result will be normally distributed even if the individual variable distributions are not.

Example

Probability of rolling a die

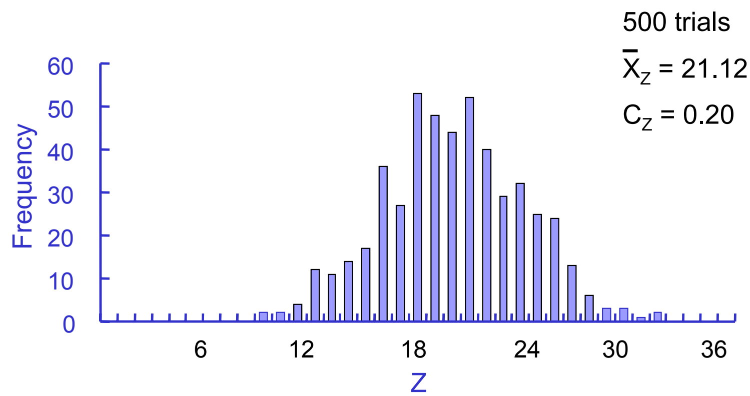


Uniform discrete distribution

Let Z be the summation of six dice

$$Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

Results



Central limit theorem states that the result should be normal for large n

Central Limit Theorem

Sums: $Z = X_1 \pm X_2 \pm X_3 \pm X_4 \pm ..... X_n$

$Z \rightarrow$ Normal as n increases

Products: $Z = X_1 \bullet X_2 \bullet X_3 \bullet X_4 \bullet ..... X_n$

$Z \rightarrow$ LogNormal as n increases

Normal and LogNormal distributions are often employed for analysis even though the underlying population distribution is unknown.

Key Points

- All variables are random and can be characterized by a statistical distribution with a mean and variance.
- The final result will be normally distributed even if the individual variable distributions are not.

Probabilistic Aspects of Fatigue

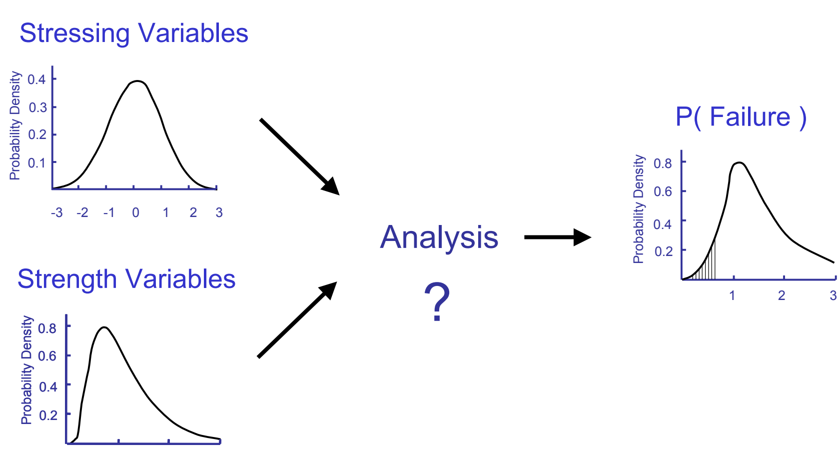
Analysis Methods

Professor Darrell F. Socie
Department of Mechanical and Industrial Engineering

Probabilistic Aspects of Fatigue

- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- **Analysis Methods**
- Characterizing Variability
- Case Studies
- FatigueCalculator.com
- GlyphWorks

Reliability Analysis



Probabilistic Analysis Methods

- Monte Carlo
  - Simple
  - Hypercube sampling
  - Importance sampling
- Analytical
  - First order reliability method FORM
  - Second order reliability method SORM

Limit States

- Limit States
  - Equilibrium
  - Strength
  - Deformation
  - Wear
  - Functional

Limit State Problems

Response function

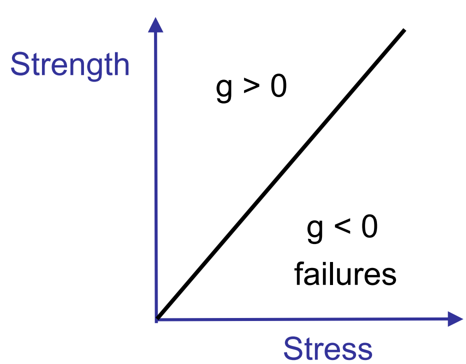
$Z(X) = Z(Q, L, b, h)$

Limit State function

$g = Z(X) - Z_o = 0$

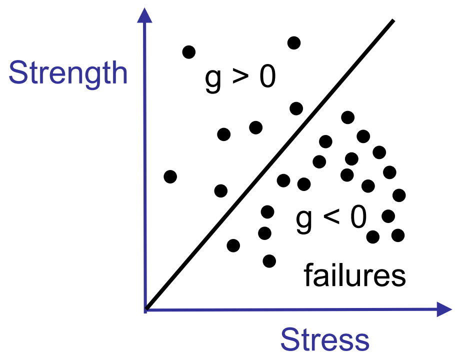
Same as the failure function

Limit States



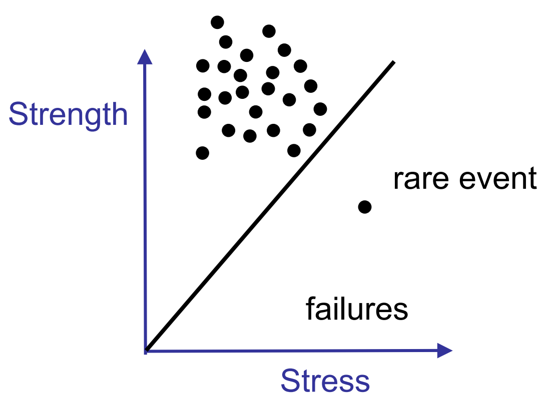
$g = \text{Strength} - \text{Stress}$

Monte Carlo



Monte Carlo is simple but what if each of these calculations required a separate FEM model?

Low Probabilities of Failure



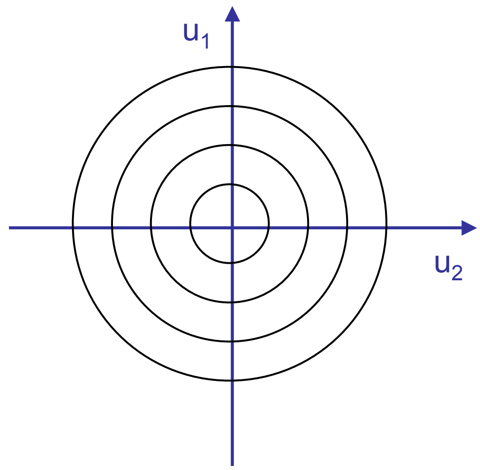
Need about 10^5 simulations for P(Failure) = 10^{-4}

Transform Variables

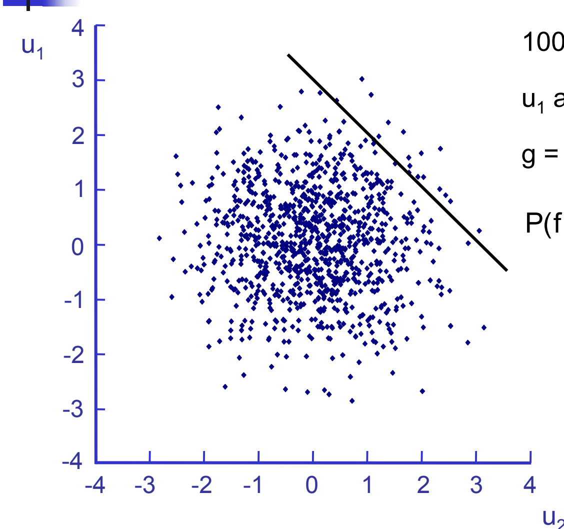
$$u_1 = \frac{x - \mu_x}{\sigma_x}$$

$\mu_{u1} = 0$

$\sigma_{u1} = 1$



Standard Monte Carlo



1000 trials

u_1 and u_2 normally distributed

$g = 3 – (u_1 + u_2)$

$$P(f) = \frac{N_f}{N} = 0.018$$

Multiple Simulations

1	25
2	15
3	19
4	21
5	16
6	22
7	16
8	12
9	15
10	11
11	19
12	15
13	20
14	17
15	9
16	23
17	15
18	20
19	14
20	21
21	16
22	18
23	22
24	13
25	20

1000 trials

25 simulations

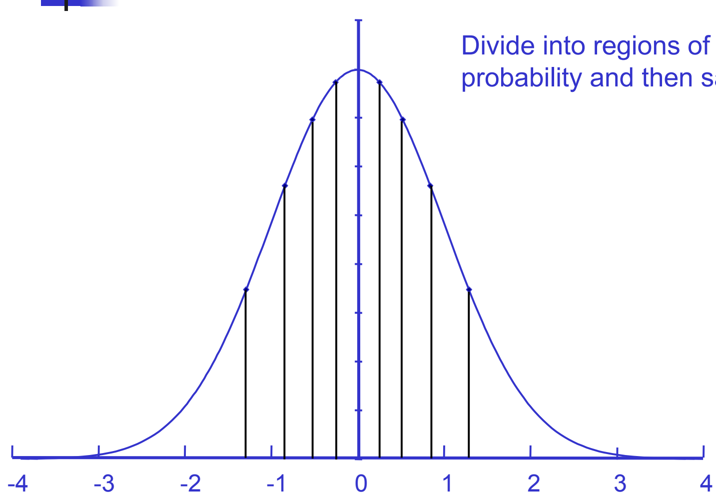
25,000 calculations

μ = 0.017

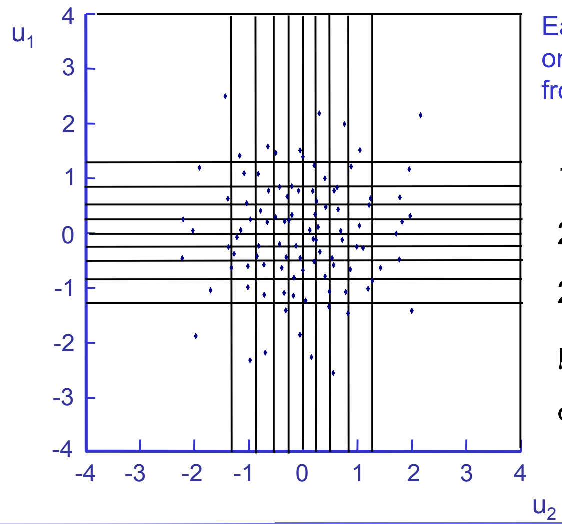
σ = 0.0040

Stratified Sampling Methods

Divide into regions of constant probability and then sample by region



Stratified Sample



Each box should have one sample drawn at random from the underlying pdf

100 trials

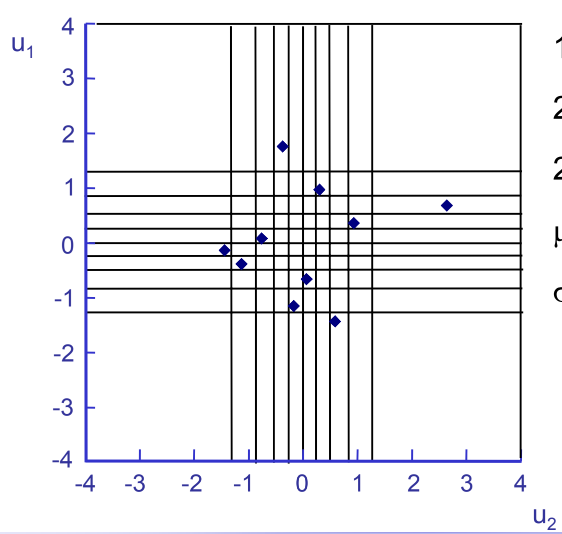
25 simulations

2,500 calculations

$\mu = 0.02$

$\sigma = 0.0093$

Latin Hypercube Sample



10 trials

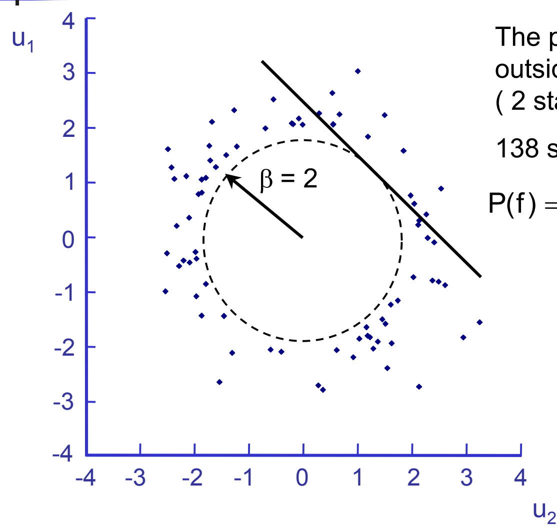
25 simulations

250 calculations

$\mu = 0.017$

$\sigma = 0.038$

Importance Sampling (continued)



The probability of a sample outside the circle is $P_S = 0.14$ (2 standard deviations)

138 simulations

$$P(f) = \frac{N_f}{N_S} P_S = 0.0183$$

Analytical Methods Strategy

- Develop a response function
- Transform the set of N variables to a set of uncorrelated u variables with μ=0 and σ=1
- Locate the most likely failure point
- Approximate the integral of the joint probability distribution over the failure region

Analytical Methods Outline

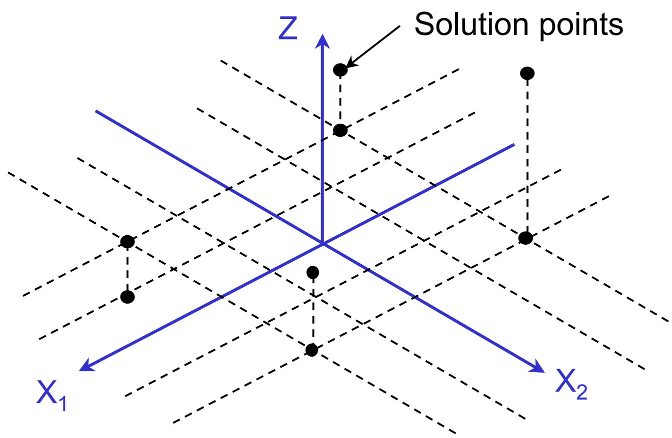
- Response Surface
- Transform Variables
- Limit State Concept
- Most Probable Point
- Probability Integration
  - FORM
  - SORM

Response Surface Methodology

Response surface methodology is a well established collection of mathematical and statistical techniques for applications where the response of interest is influenced by several variables.

$$Z(X) = f(x_1, x_2, x_3, \cdots x_n) + \varepsilon$$

Response Surface



Fit a surface through the points

Mathematical Representation

$$Z(X) = A_o + A_1x_1 + A_2x_2 + A_3x_3 + \cdots\cdots \qquad \text{Linear}$$

$$B_1x_1^2 + B_2x_2^2 + B_3x_3^2 + \cdots\cdots \qquad \text{Incomplete Quadradic}$$

$$C_1x_1x_2 + C_2x_1x_3 + C_3x_2x_3 + \cdots \qquad \text{Complete Quadradic}$$

	Solutions
Linear	N + 1
Incomplete Quadradic	2N + 1
Complete Quadradic	N(N + 1)/2

Evaluation of Response Surface

Suppose stress is affected by speed and temperature

Factorial Design

speed \ temperature	high	low
high	X	X
low	X	X

4 deterministic solutions needed

2^n solutions

Evaluation of Response Surface

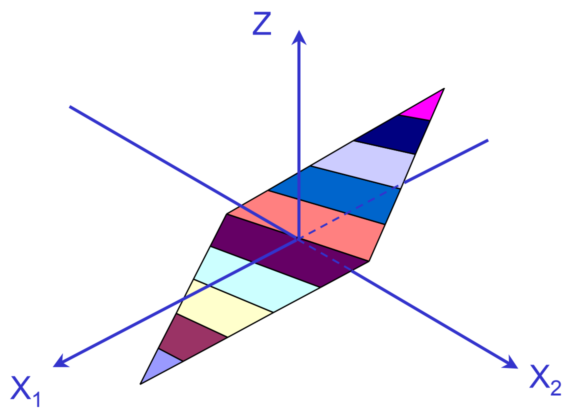
Factorial Design

speed \ temperature	high		low
high	X	X	X
	X	X	X
low	X	X	X

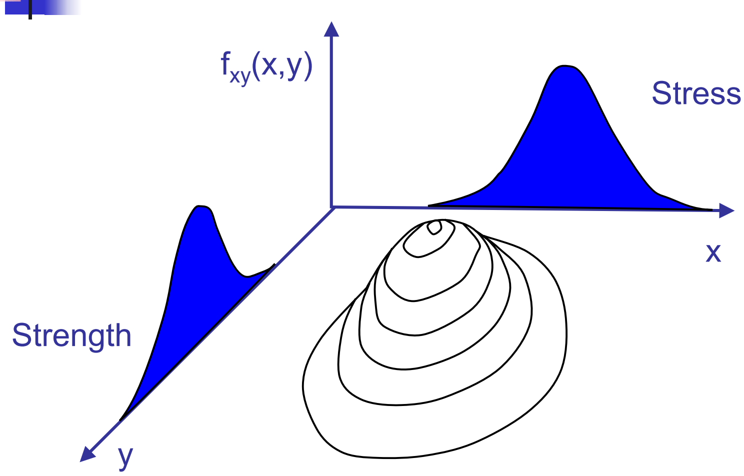
9 deterministic solutions needed

3^n solutions

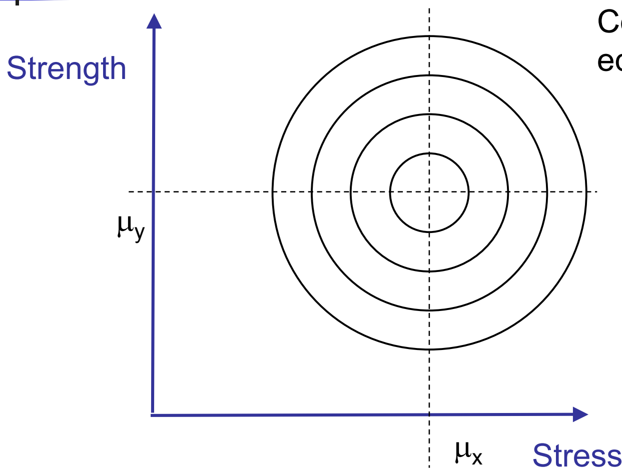
Graphical Representation



Joint Probability Density



2D - Joint Probability Density



Contours of equal probability

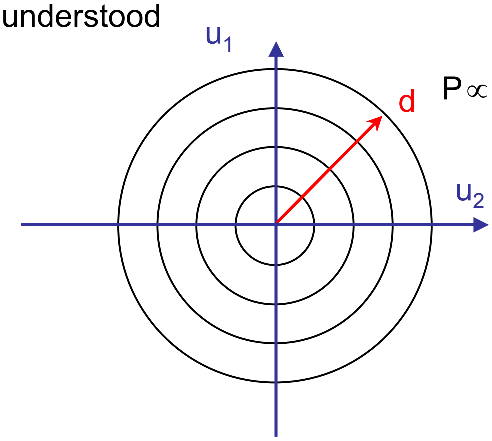
Transform Variables

The mathematics and behavior of normal distributions are well understood

$$u_1 = \frac{x - \mu_x}{\sigma_x}$$

$\mu_u = 0$

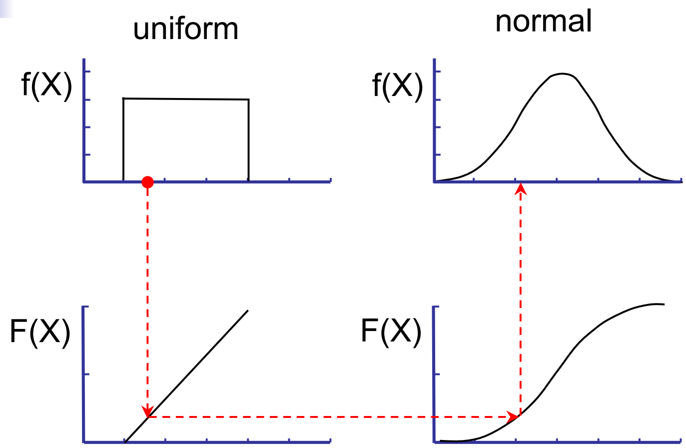
$\sigma_u = 1$



$$P \propto \exp\left(-0.5\, d^2\right)$$

Any distribution can be mapped into a normal distribution

X to u mapping



Limit States

Limit states define probability problems

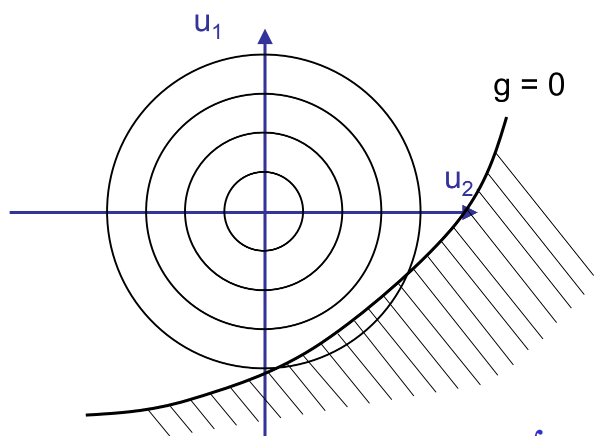
For example:

$$g = \text{Strength}(x_1) - \text{Stress}(x_2)$$

$$\text{Prob}(\ g <= 0\ ) = \text{Probability of failure} = P_f$$

Analysis focuses on $g = 0$

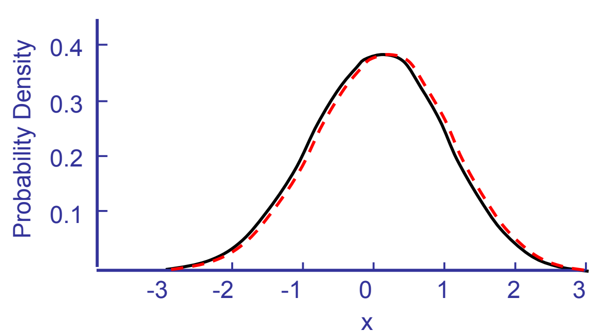
Failure Probability



$$P(\text{Failure}) = \int_{g<0} f_{u_1 u_2}(u_1, u_2)\, du_1\, du_2$$

Thought Experiment

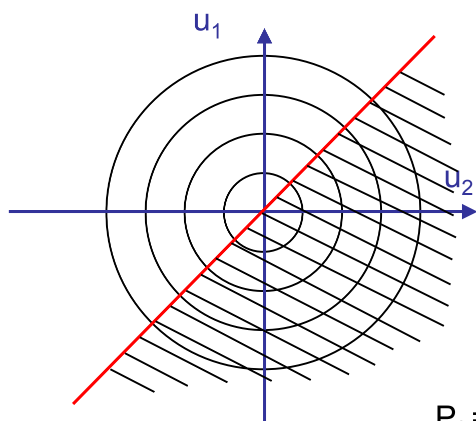
Suppose stress and strength had the same distribution



What is the probability of failure?

g function

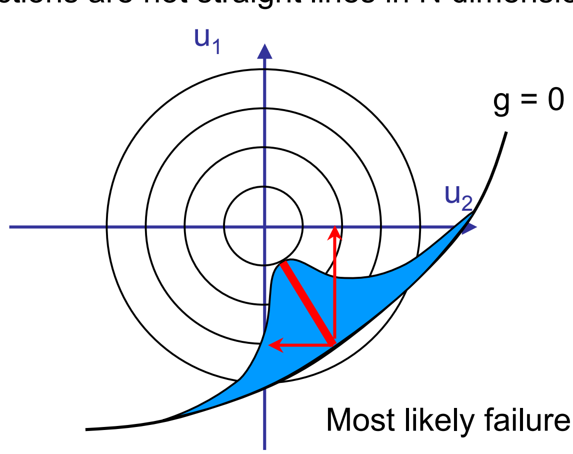
$g = \text{Strength}(x_1) - \text{Stress}(x_2)$



$P_f = 50\%$

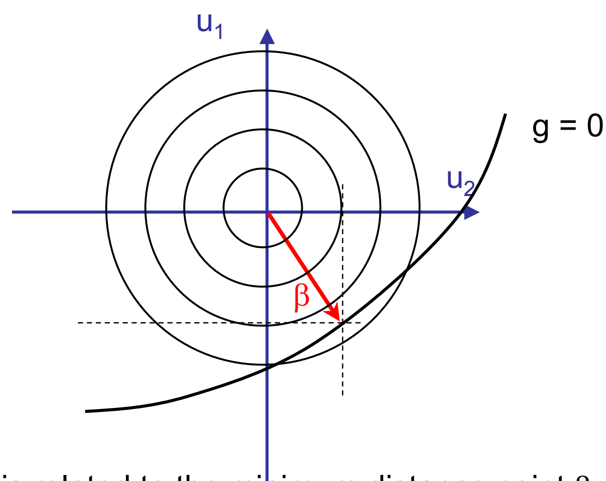
Probable Combinations of u_1 and u_2

g functions are not straight lines in N dimensional space



Most likely failure point

Most Probable Point



Probability is related to the minimum distance point β

Safety Index β

Sometimes called reliability index

β is a number measured in standard deviations

Unlike safety factors, failure probability is directly related to the safety index

$$P_f = \Phi(-\beta)$$

Determining the MPP

Requires an efficient numerical search to find the tangent point of a hypersphere (β-sphere) and the limit state function in **u** space

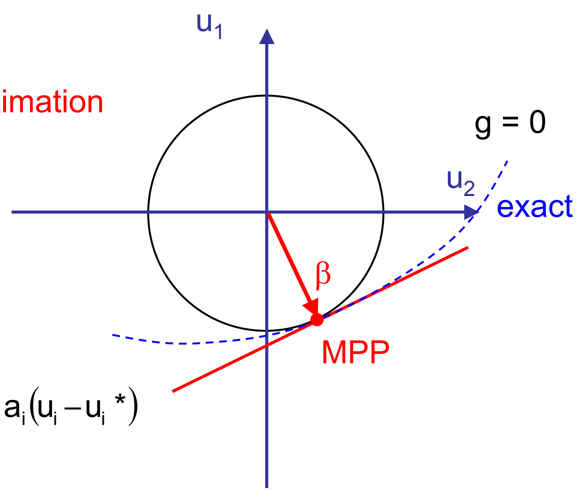
Probability Integration

- FORM
- SORM

First Order Reliability Model

FORM

Linear approximation

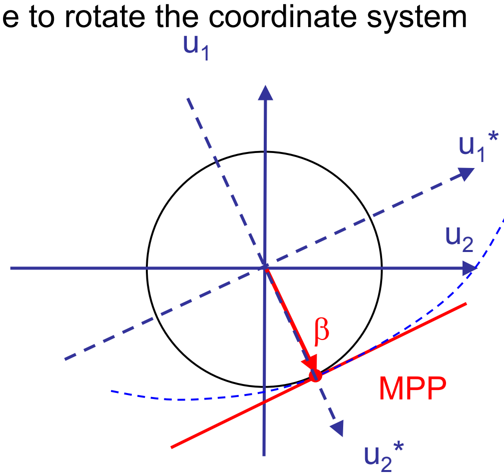


$$g(u) = a_o + \sum_{i=1}^{n} a_i (u_i - u_i^*)$$

FORM

The joint probability density is rotationally symmetric so it is possible to rotate the coordinate system

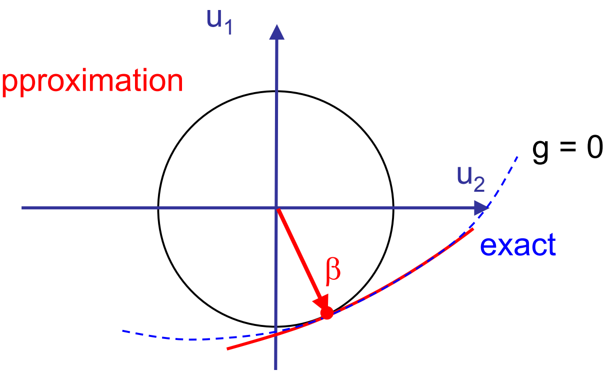
$$P_f \approx \Phi(-\beta)$$



Second Order Reliability Model

SORM

Quadradic approximation

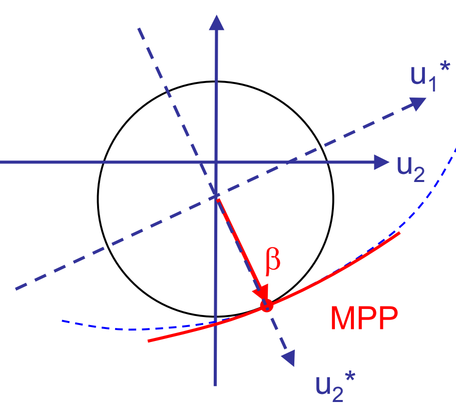


$$g(u) = a_o + \sum_{i=1}^{n} a_i (u_i - u_i^*) + \sum_{i=1}^{n} b_i (u_i - u_i^*)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} (u_i - u_i^*)(u_j - u_j^*)$$

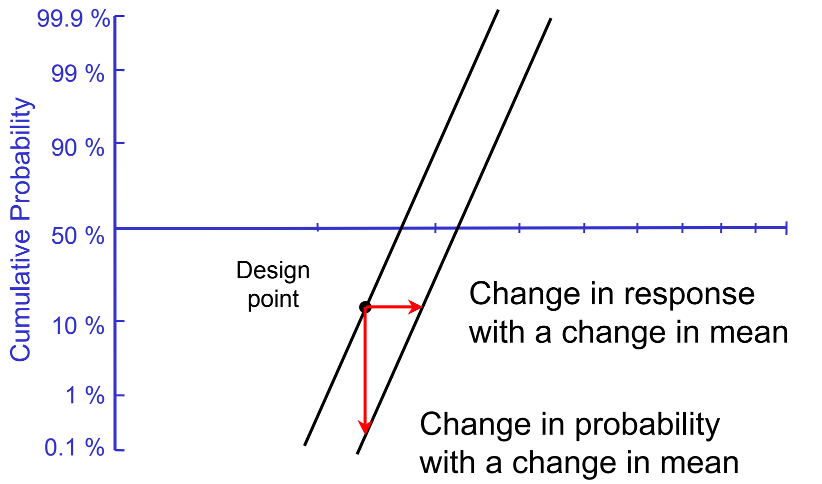
SORM

$$P_f = \Phi(-\beta) \prod_{i=1}^{n} (1 - \beta \kappa_i)$$

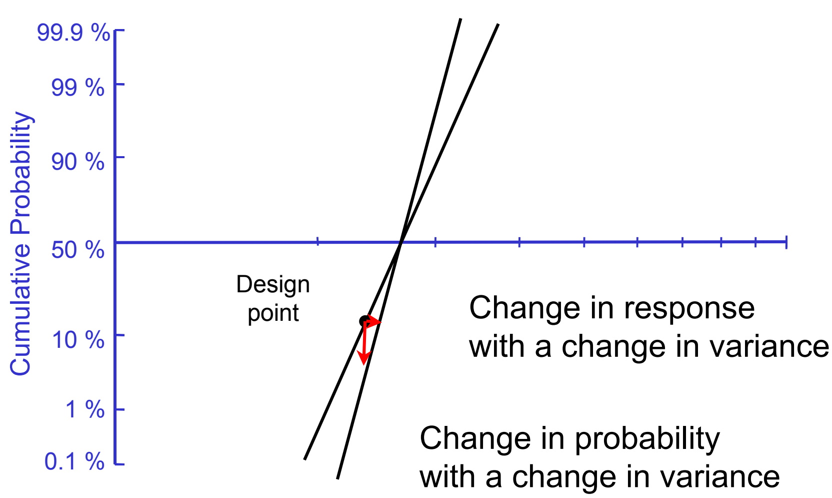
κ_i surface curvature



Sensitivity Factors



Sensitivity Factors



Deterministic Sensitivity Factors

$$\frac{\partial Z(X)}{\partial X_i}$$

Change in response mean with respect to a change in the input variable mean

Frequently normalized by the means to compare relative importance of variables

$$\frac{\partial Z(X)/\overline{Z}}{\partial X_i/\overline{X}}$$

Probabilistic Sensitivity Coefficient

Standard deviation sensitivity

$$S_{\sigma i} = \frac{\partial P/P}{\partial \sigma_i/\sigma_i}$$

Mean deviation sensitivity

$$S_{\mu i} = \frac{\partial P/P}{\partial \mu_i/\mu_i}$$

Probabilistic Sensitivity Factors

$$\alpha_i \propto \left(\frac{\partial Z(X)}{\partial X_i} \right) \sigma_i \qquad \sum \alpha_i^2 = 1$$

α_i - probabilistic sensitivity
$Z(X)$ - response
X_i - input variable
σ_i - standard deviation of X_i

Software

- General Purpose
- Durability

Probabilistic Aspects of Fatigue

Variability

Professor Darrell F. Socie
Department of Mechanical and Industrial Engineering

Probabilistic Aspects of Fatigue

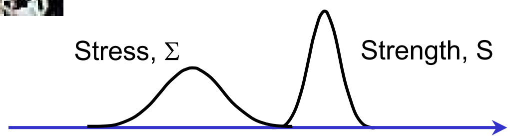
- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- Analysis Methods
- **Characterizing Variability**
- Case Studies
- FatigueCalculator.com
- GlyphWorks

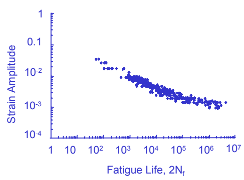
Sources of Variability

customers ← Stress → usage

Stress, Σ

Strength, S





materials ← Strength → manufacturing

Variability and Uncertainty

Variability: Every apple on a tree has a different mass.

Uncertainty: The variety of the apple is unknown.

Variability: Fracture toughness of a material

Uncertainty: The correct stress intensity factor solution

Sources of Variability

- Stress Variables
  - Loading
  - Customer Usage
  - Environment
- Strength Variables
  - Material
  - Processing
  - Manufacturing Tolerance
  - Environment

Sources of Uncertainty

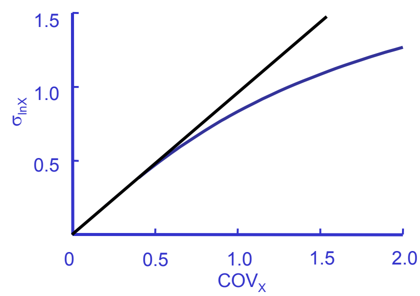
- Statistical Uncertainty
  - Incomplete data (small sample sizes)
- Modeling Error
  - Analysis assumptions
- Human Error
  - Calculation errors
  - Judgment errors

Modeling Variability

Central Limit Theorem:

Products: $Z = X_1 \bullet X_2 \bullet X_3 \bullet X_4 \bullet \ldots.. X_n$

$Z \rightarrow$ LogNormal as n increases



$\sigma_{lnX} \sim COV_X$

COV_X is a good measure of variability

COV and LogNormal Distributions

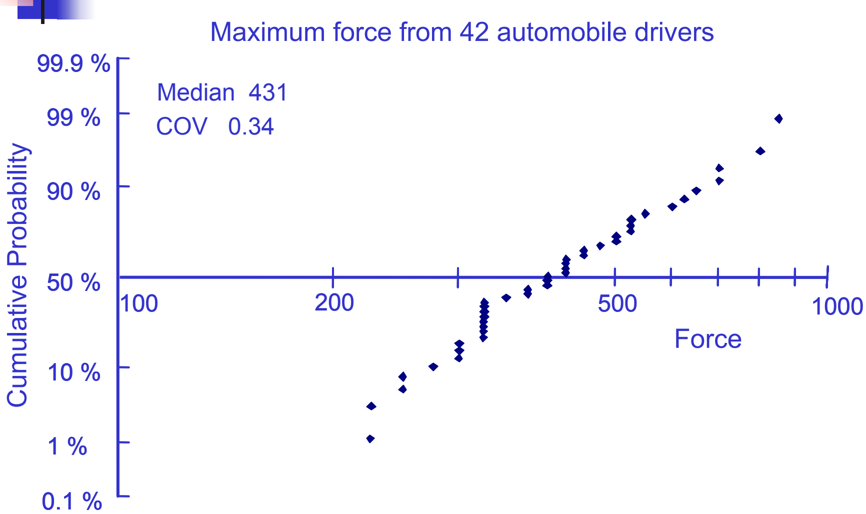
COV_X	Standard Deviation, lnx		
	1 68.3%	2 95.4%	3 99.7%
0.05	1.05	1.11	1.16
0.1	1.10	1.23	1.33
0.25	1.28	1.66	2.04
0.5	1.60	2.64	3.92
1	2.30	5.53	11.1

99.7% of the data is within a factor of ± 1.33 of the mean for a COV = 0.1

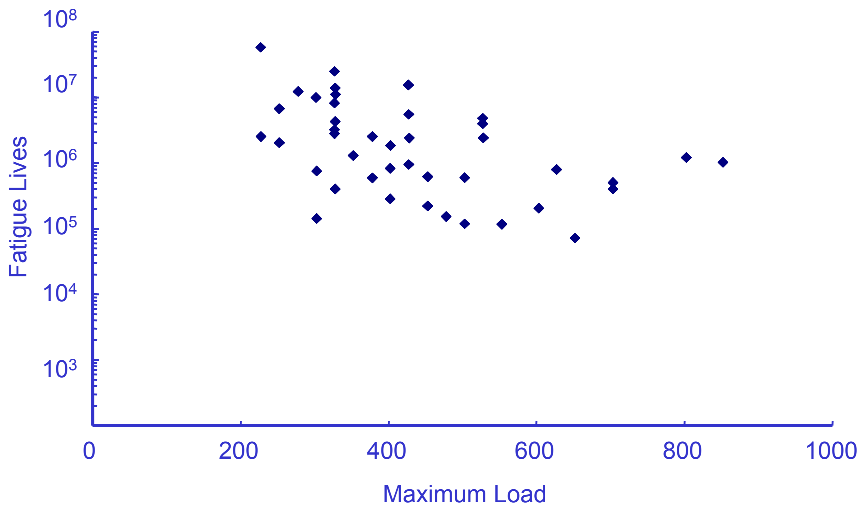
Variability in Service Loading

- Quantifying Loading Variability
  - Maximum Load
  - Load Range
  - Equivalent Stress

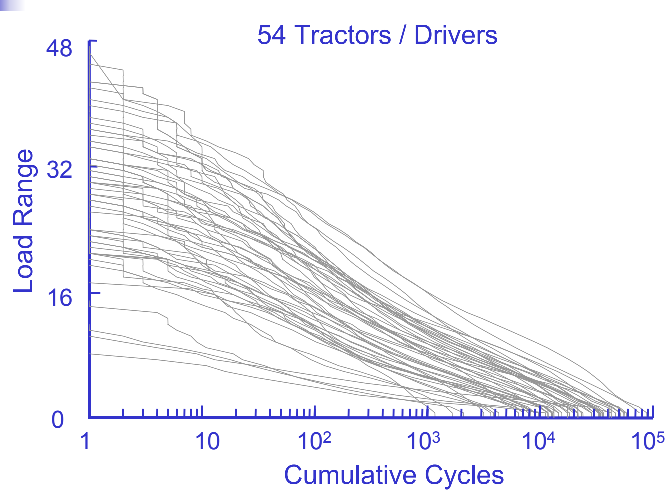
Maximum Force



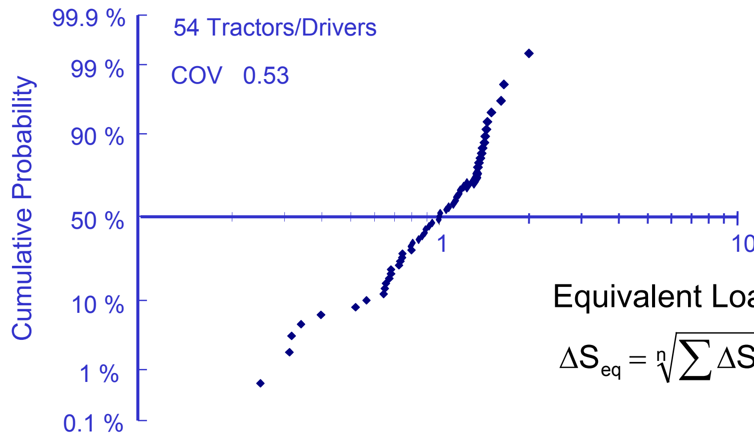
Maximum Load Correlation



Loading Variability



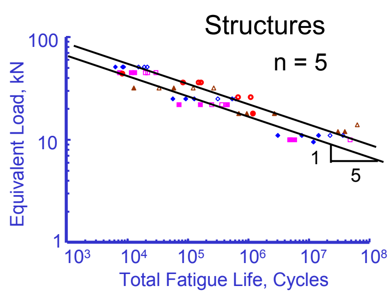
Variability in Loading

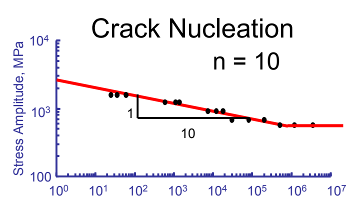


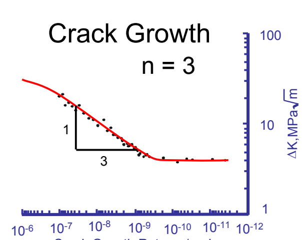
Equivalent Load

$$\Delta S_{eq} = \sqrt[n]{\sum \Delta S^n}$$

Mechanisms and Slopes

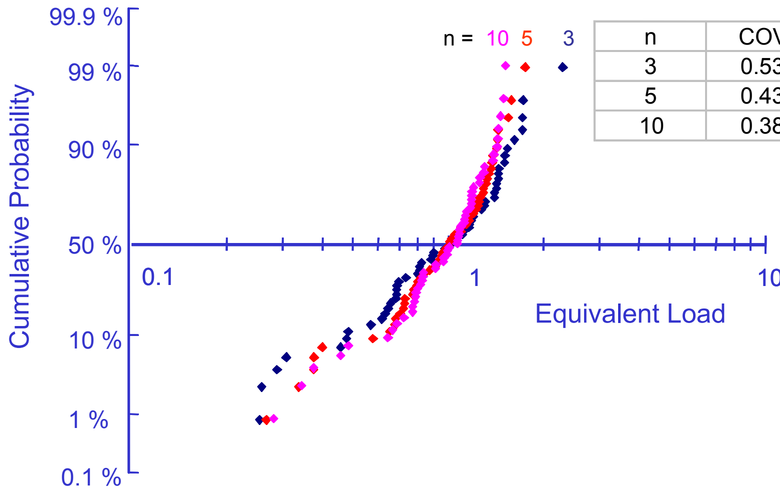






A combination of nucleation and growth

Effect of Slope on Variability

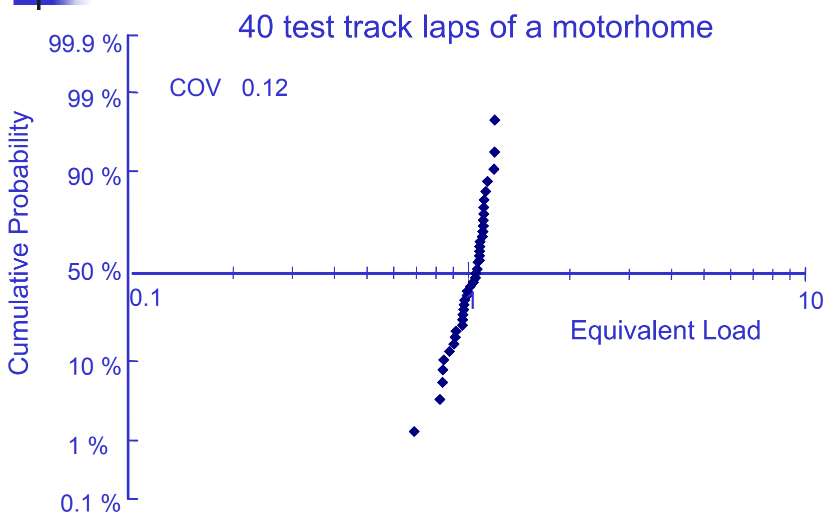


n	COV
3	0.53
5	0.43
10	0.38

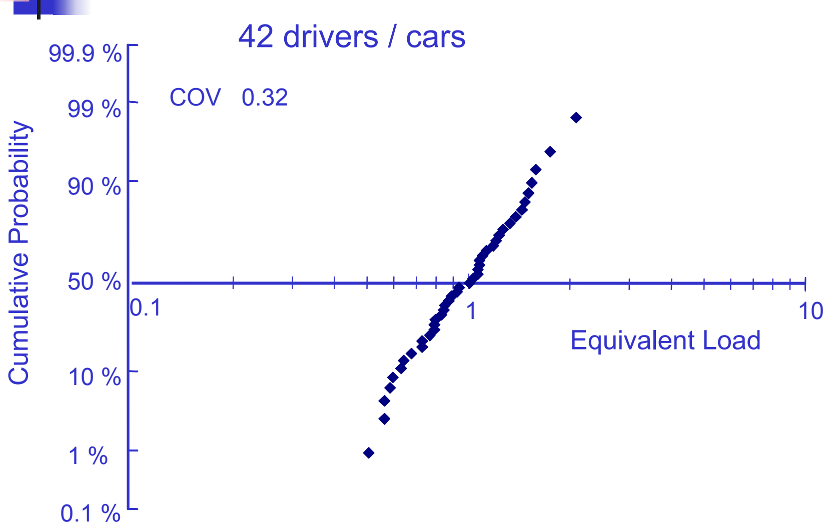
Loading History Variability

- Test Track
- Customer Service

Test Track Variability



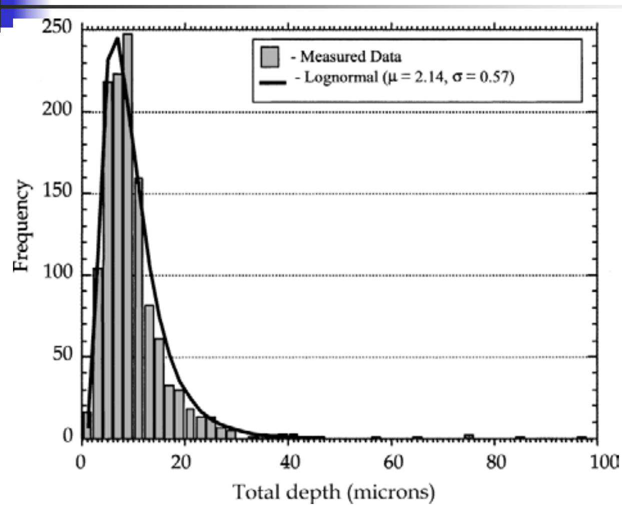
Customer Usage Variability



Variability in Environment

- Inclusions
- Pit depth

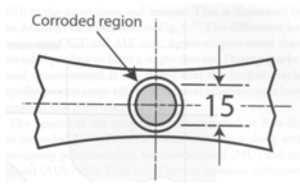
Inclusions That Initiated Cracks

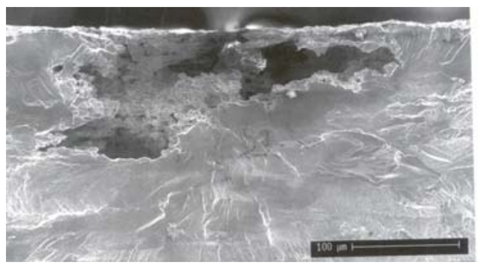


COV = 0.27

Barter, S. A., Sharp, P. K., Holden, G. & Clark, G. "Initiation and early growth of fatigue cracks in an aerospace aluminium alloy", *Fatigue & Fracture of Engineering Materials & Structures* **25** (2), 111-125.

Pits That Initiated Cracks





7010-T7651

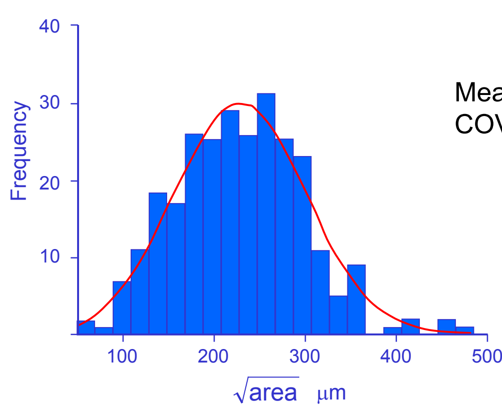
Pre-corroded specimens

300 specimens

246 failed from pits

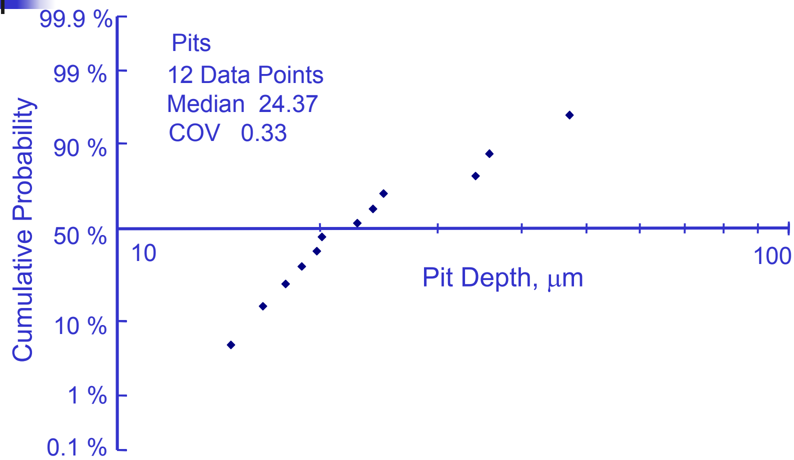
Crawford et.al."The EIFS Distribution for Anodized and Pre-corroded 7010-T7651 under Constant Amplitude Loading" *Fatigue and Fracture of Engineering Materials and Structures*, Vol. 28, No. 9 2005, 795-808

Pit Size Distribution



Mean = 230
COV = 0.32

Pit Depth Variability

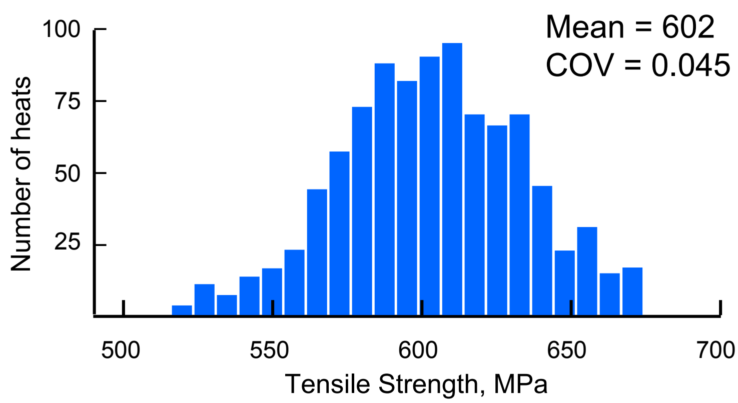


Dolly, Lee, Wei, “The Effect of Pitting Corrosion on Fatigue Life”
Fatigue and Fracture of Engineering Materials and Structures, Vol. 23, 2000, 555-560

Variability in Materials

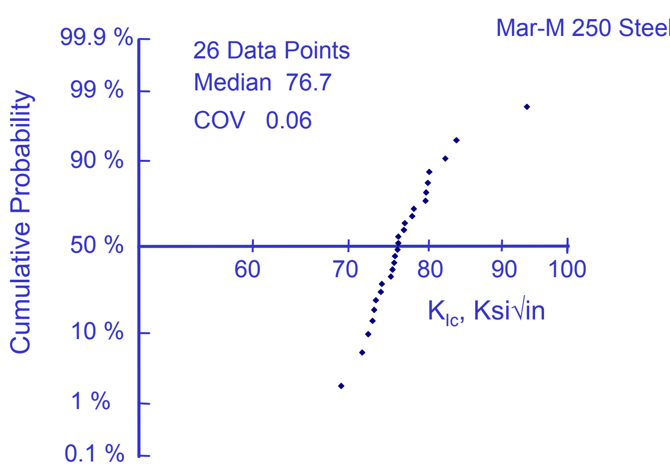
- Tensile Strength
- Fracture Toughness
- Fatigue
  - Fatigue Strength
  - Fatigue Life
- Strain-Life
- Crack Growth

Tensile Strength - 1035 Steel



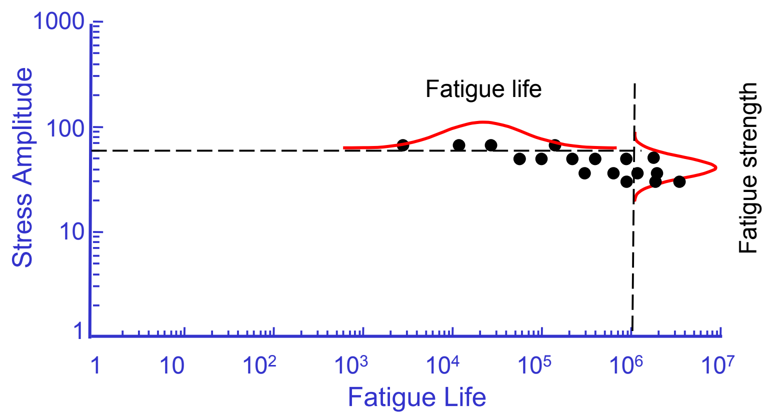
Metals Handbook, 8th Edition, Vol. 1, p64

Fracture Toughness

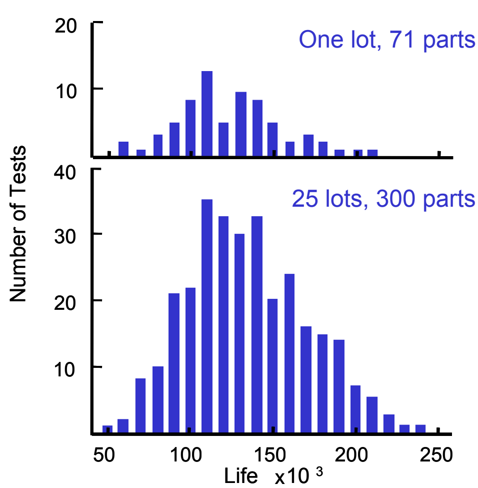


Kies, J.A., Smith, H.L., Romine, H.E. and Bernstein, H, “Fracture Testing of Weldments”, ASTM STP 381, 1965, 328-356

Fatigue Variability



Fatigue Life Variability



Production torsion bars
5160H steel

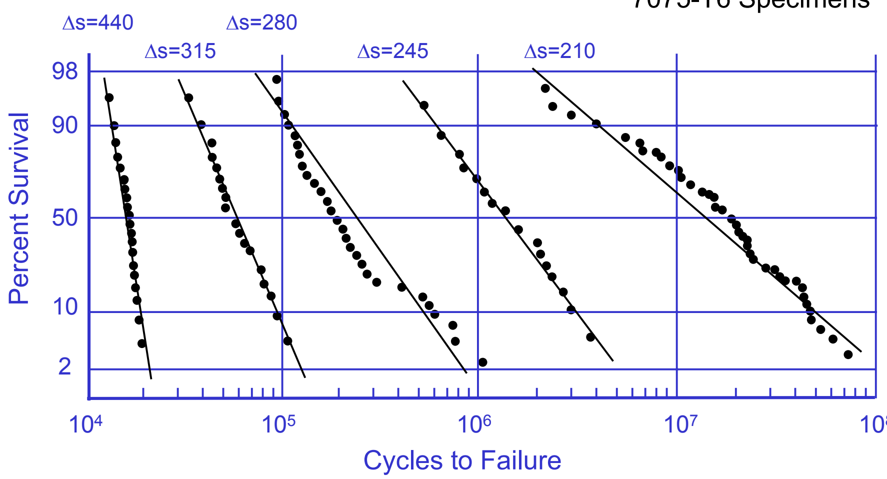
μ_x = 123,000 cycles
COV = 0.25

μ_x = 134,000 cycles
COV = 0.27

Metals Handbook, 8th Edition, Vol. 1, p219

Statistical Variability of Fatigue Life

7075-T6 Specimens



Sinclair and Dolan, “Effect of Stress Amplitude on the Variability in Fatigue Life of 7075-T6 Aluminum Alloy” Transactions ASME, 1953

COV vs Fatigue Life

ΔS	$\overline{X}$	COV
440	14,000	0.12
315	25,000	0.38
280	220,000	0.70
245	1,200,000	0.67
210	12,000,000	1.39

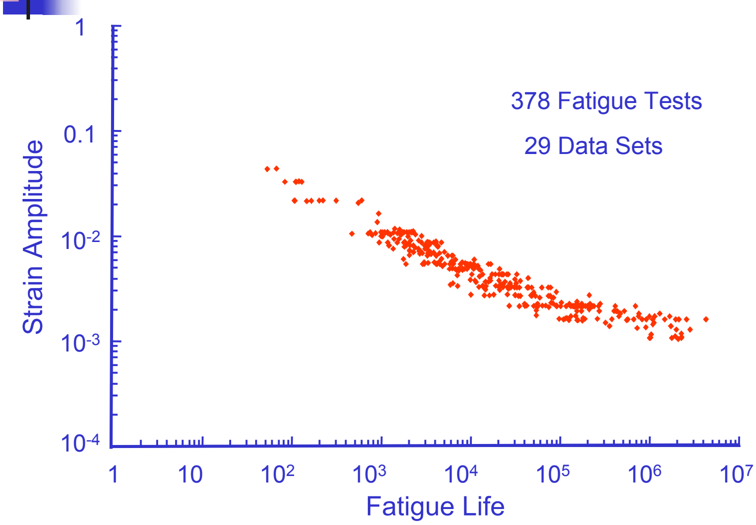
Variability in Fatigue Strength

$$\frac{\Delta S}{2} = S_f^{'} (N_f)^b \quad b \approx -0.085$$

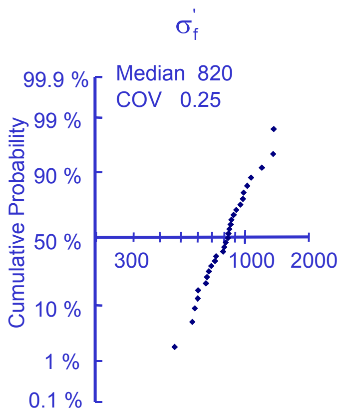
$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n} \left(1 + C_{X_i}^{\ 2}\right)^{a_i^2} - 1}$$

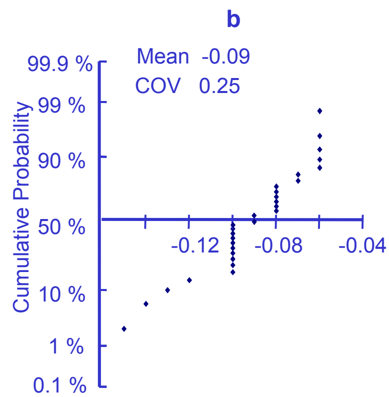
$$C_{S_f^{'}} = \sqrt{\left(1+1.39^2\right)^{(-.085)^2} - 1} = 0.088$$

Strain Life Data for 950X Steel

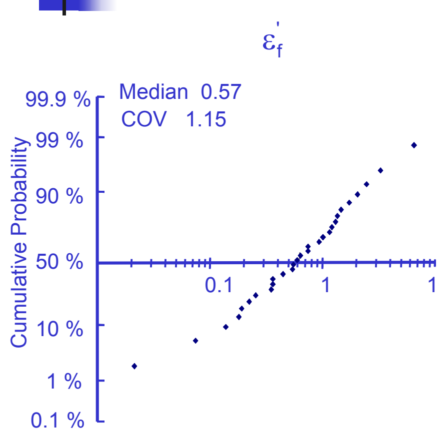


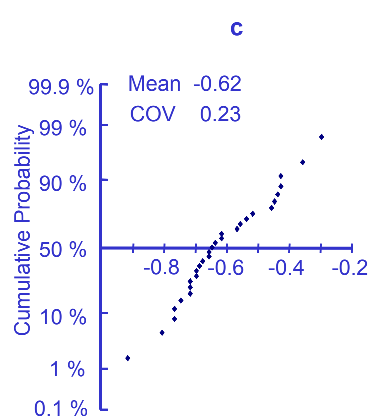
29 Individual Data Sets





29 Individual Data Sets (continued)

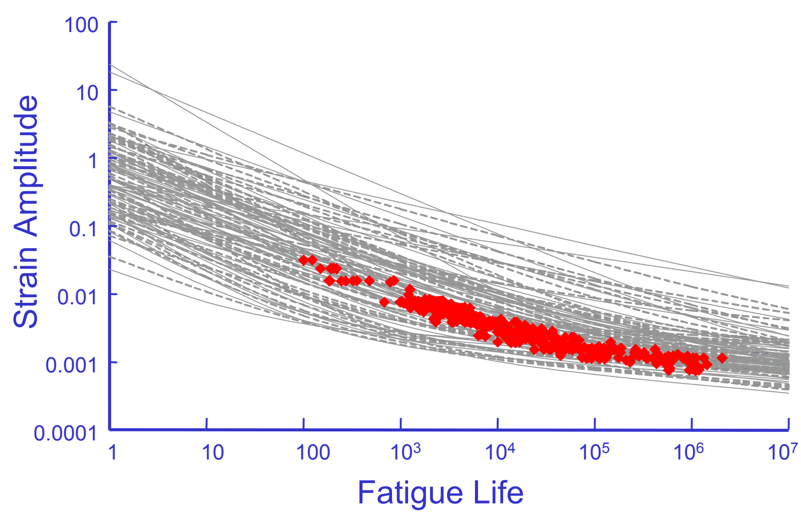




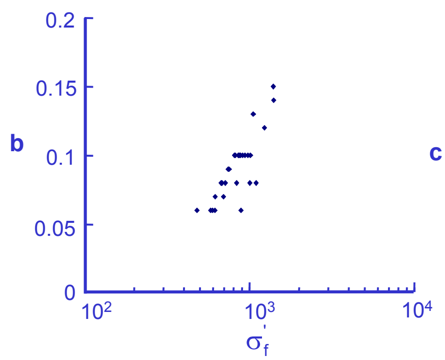
Input Data Simulation

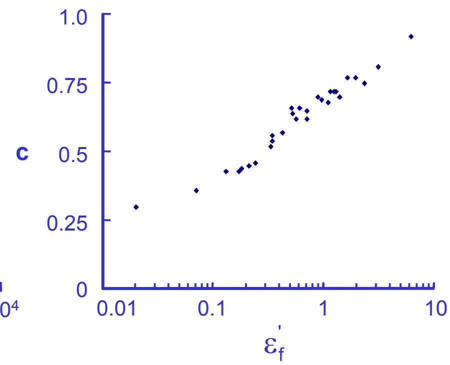
$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'\left(L, \mu_{\sigma_f}, \sigma_{\sigma_f}\right)}{E}\left(2N_f\right)^{b(N, \mu_b, \sigma_b)} + \varepsilon_f'\left(L, \mu_{\varepsilon_f}, \sigma_{\varepsilon_f}\right)\left(2N_f\right)^{c(N, \mu_b, \sigma_b)}$$

Simulation Results



Correlation





ρ = -0.828

ρ = -0.976

Generating Correlated Data

$$z_1 = \Phi(\text{ rand() }) \qquad z_1 = N(0,1)$$

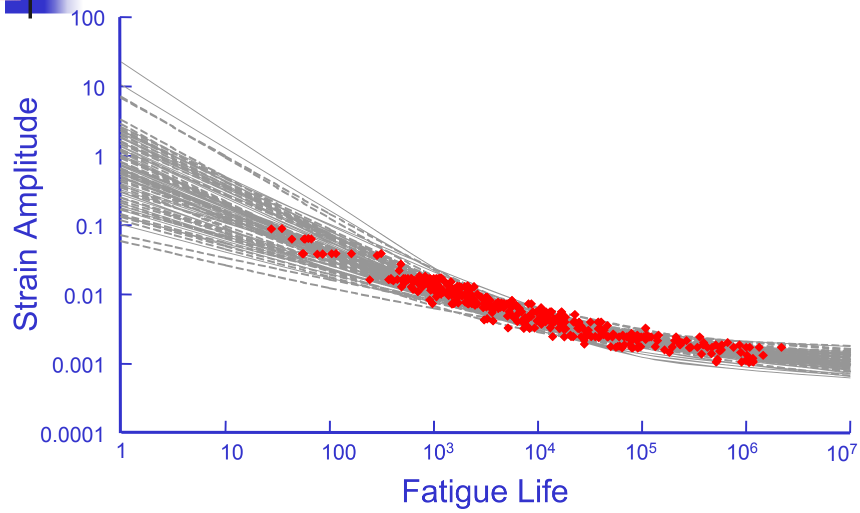
$$z_2 = \Phi(\text{ rand() })$$

$$z_3 = z_1 \rho + z_2 \sqrt{1-\rho^2}$$

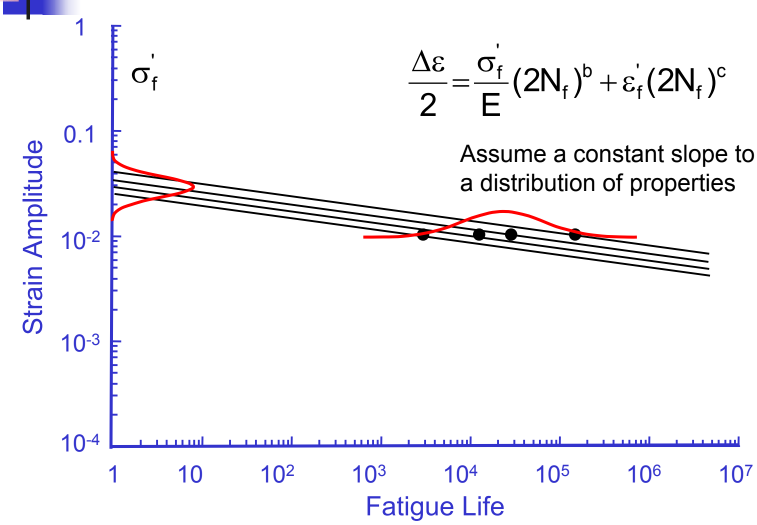
$$\sigma_f' = \exp(\mu_{\ln \sigma_f'} + \sigma_{\ln \sigma_f'} z_1)$$

$$b = \mu_b + \sigma_b z_3$$

Correlated Properties



Curve Fitting



$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'}{E}(2N_f)^b + \varepsilon_f'(2N_f)^c$$

Assume a constant slope to get a distribution of properties

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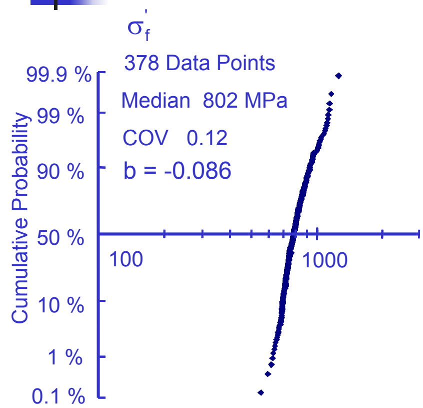
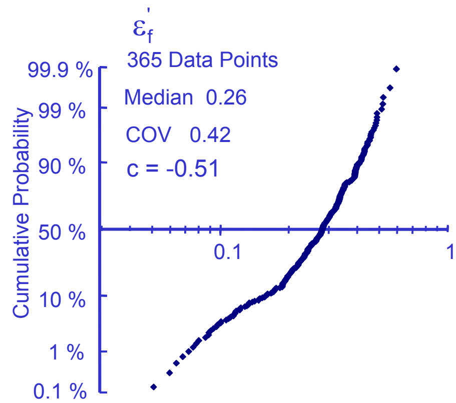
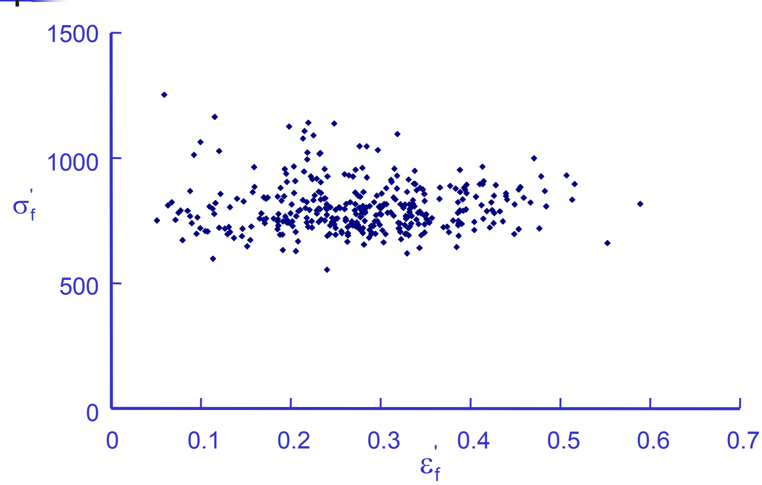
Property Distribution

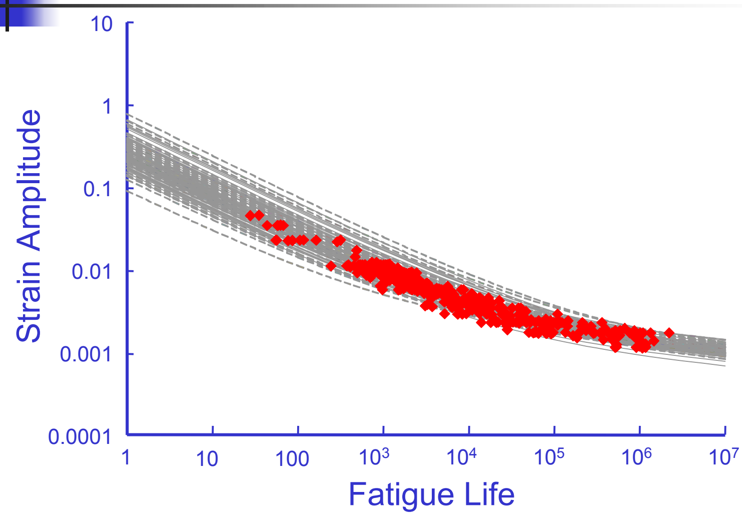




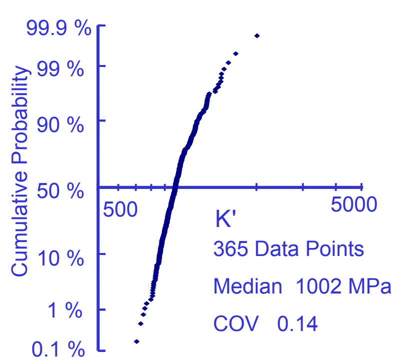
Correlation

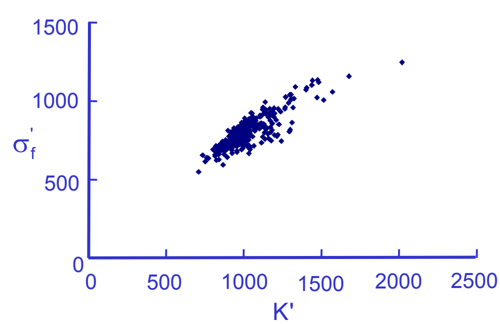


Simulation



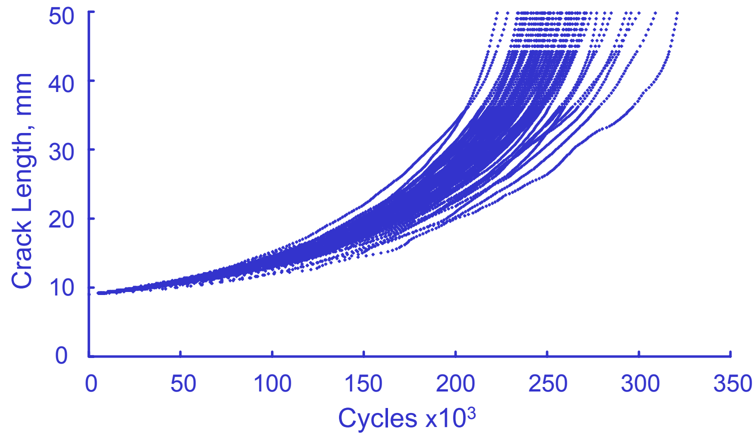
Strength Coefficient





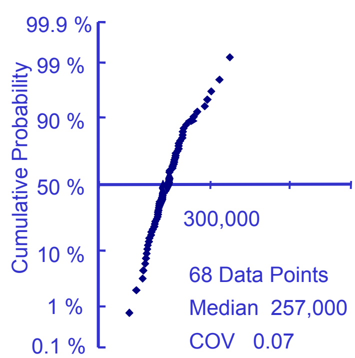
$\rho = 0.863$

Crack Growth Data

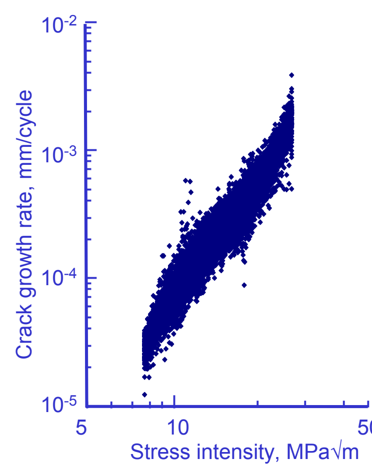


Virkler, Hillberry and Goel, "The Statistical Nature of Fatigue Crack Propagation", Journal of Engineering Materials and Technology, Vol. 101, 1979, 148-153

Crack Growth Rate Data



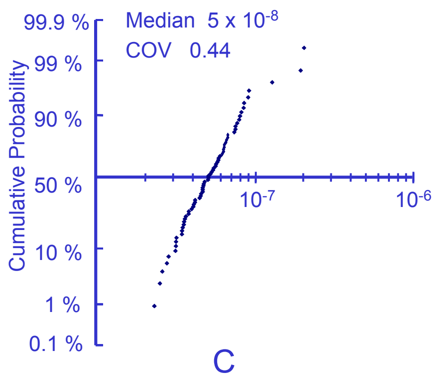
Fatigue Lives

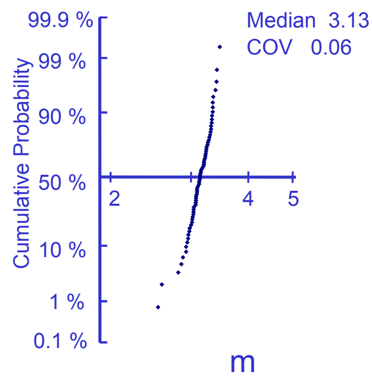


Crack Growth Rate

Crack Growth Properties

$$\frac{da}{dN} = C \Delta K^m$$





Beware of Correlated Variables

$$N_f = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C \Delta S^m \pi^{\frac{m}{2}} (1-m/2)}$$

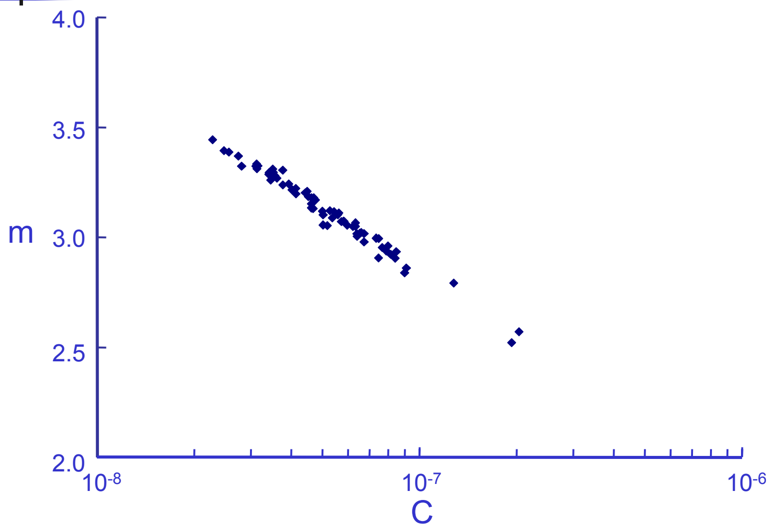
N_f and C are linearly related and should have the same variability, but

$$COV_{N_f} = 0.07$$

$$COV_C = 0.44$$

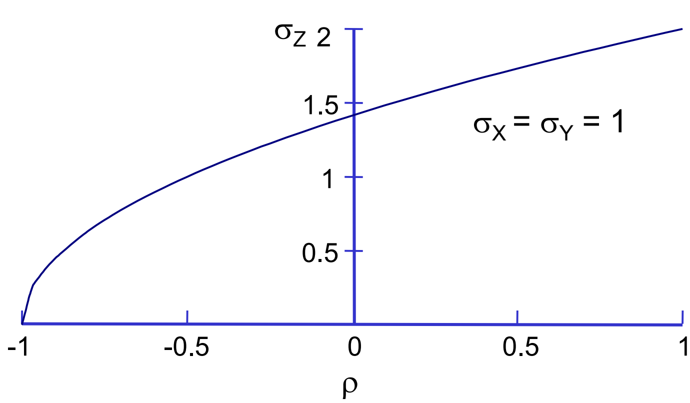
because C and m are correlated.

Correlation

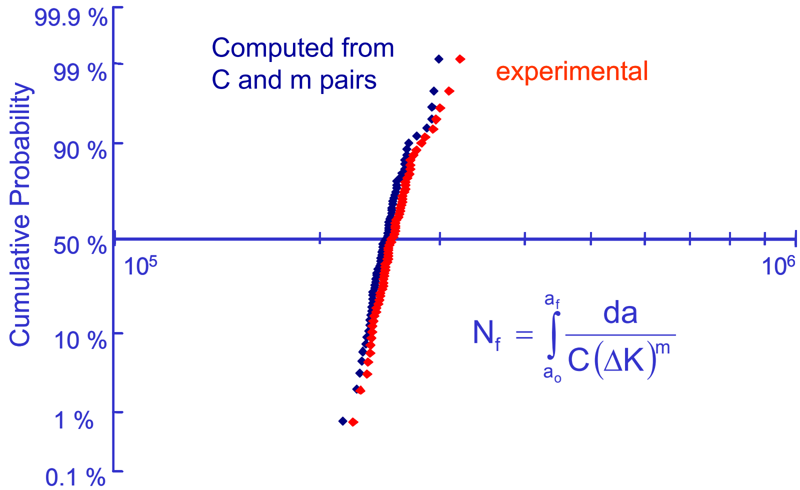


Correlated Variables

$$\sigma_Z = \sqrt{\sigma_X{}^2 + 2\rho\sigma_X\sigma_Y + \sigma_Y{}^2}$$



Calculated Lives

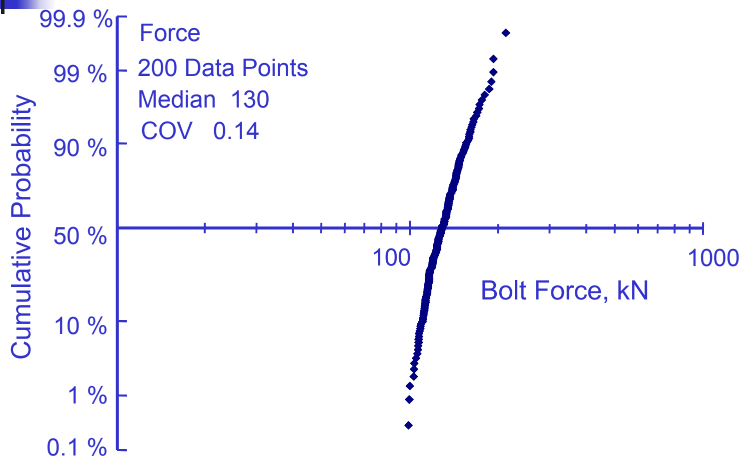


$$N_f = \int_{a_o}^{a_f} \frac{da}{C(\Delta K)^m}$$

Manufacturing/Processing Variability

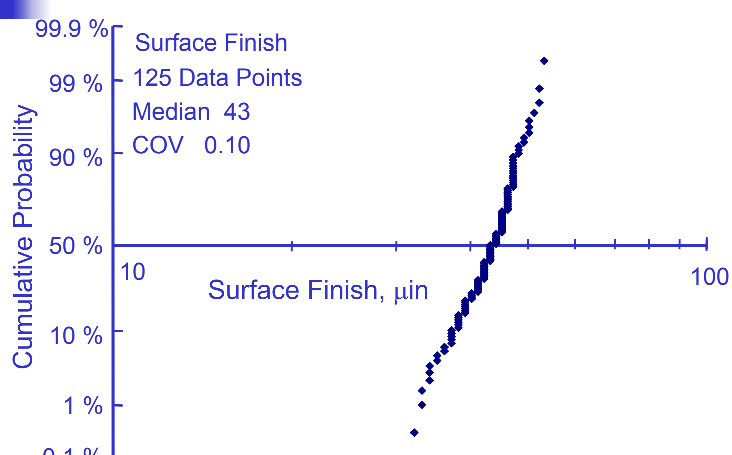
- Bolt Forces
- Surface Finish
- Drilled Holes

Variability in Bolt Force



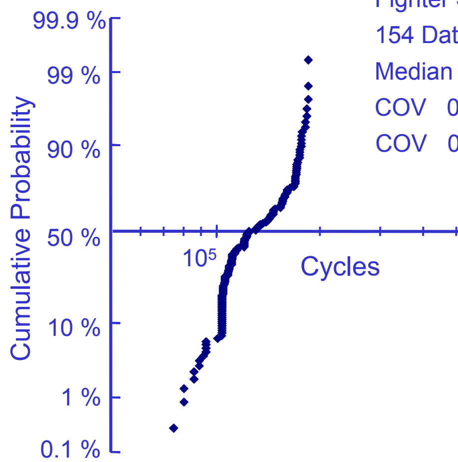
Preload force in bolts tightened to 350 Nm

Surface Roughness Variability

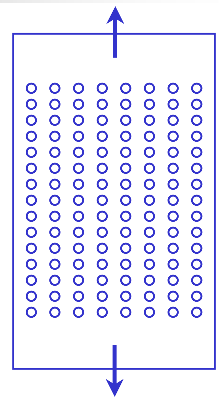


Machined aluminum casting

Drilled Holes



Fighter Spectrum
154 Data Points
Median 126,750
COV 0.22 in life
COV 0.07 in strength



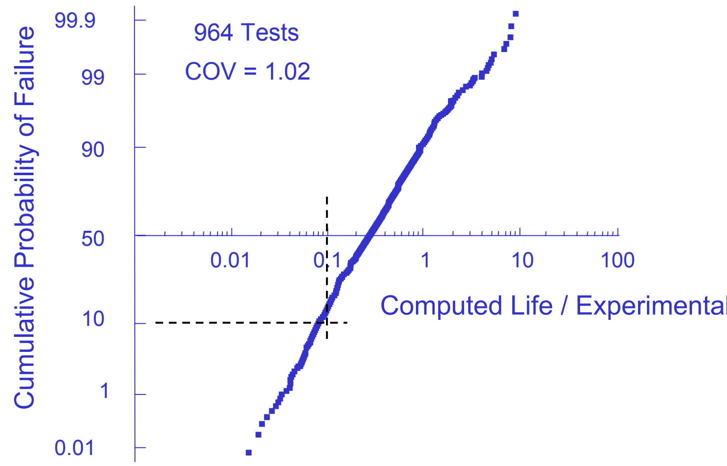
180 drilled holes in a single plate

From: J.P. Butler and D.A. Rees, "Development of Statistical Fatigue Failure Characteristics of 0.125-inch 2024-T3 Aluminum Under Simulated Flight-by-Flight Loading," ADA-002310 (NTIS no.), July 1974.

Analysis Uncertainty

- Miners Linear Damage rule
- Strain Life Analysis

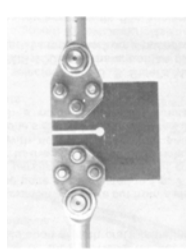
Miners Rule

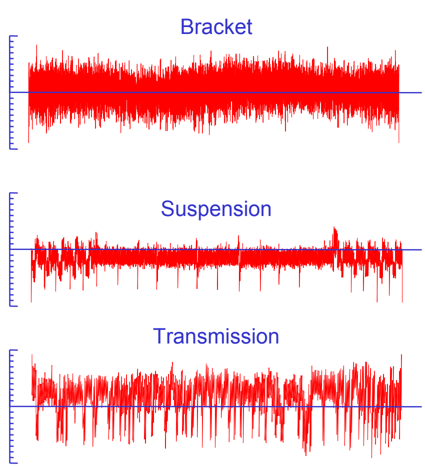


A safety factor of 10 in life would result in a 10% chance of failure

From Erwin Haibach “Betriebsfestigkeit”, Springer-Verlag, 2002

SAE Specimen

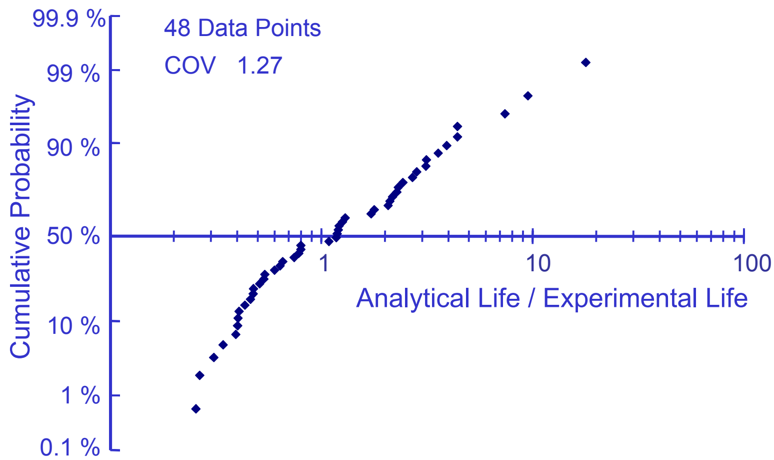




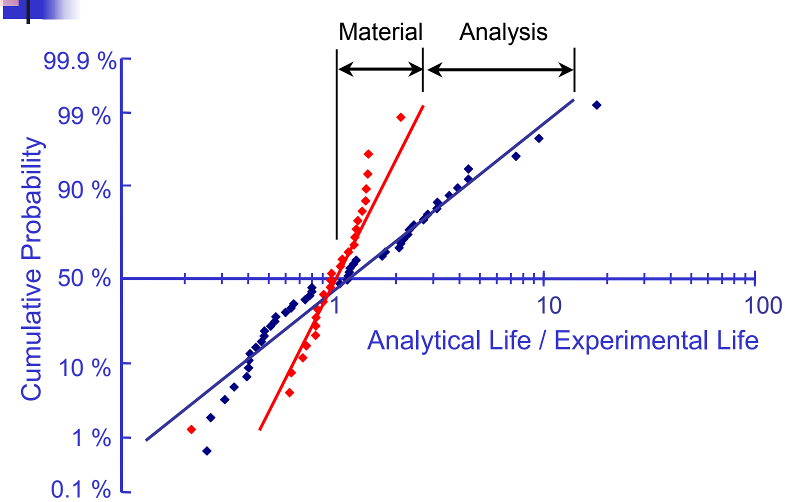
Fatigue Under Complex Loading: Analysis and Experiments, SAE AE6, 1977

Analysis Results

Strain-Life analysis of all test data



Material Variability



Strain-Life back calculation of specimen lives

Modeling Uncertainty

Analysis Uncertainty C_U = ?

The variability in reproducing the original strain life data from the material constants is $C_M \sim 0.44$

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1+C_{X_i}{}^2\right)^{a_i^2} - 1}$$

$$1+C_U{}^2 = \frac{1+C_{N_f}{}^2}{1+C_M{}^2}$$

$C_U = 1.09$

90% of the time the analysis is within a factor of 3 !
99% of the time the analysis is within a factor of 10 !

Variability from Multiple Sources

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1+C_{X_i}{}^2\right)^{a_i^2} - 1}$$

Suppose we have 4 variables each with a COV = 0.1

The combined variability is COV = 0.29

Suppose we reduce the variability of one of the variables to 0.05

The combined variability is now COV = 0.27

If all of the COV's are the same, it doesn't do any good to reduce only one of them, you must reduce all of them !

Variability from Multiple Sources

$$\text{COV} \quad C = \sqrt{\prod_{i=1}^{n}\left(1 + C_{X_i}^{\ 2}\right)^{a_i^2} - 1}$$

Suppose we have 3 variables each with a COV = 0.1 and one with COV = 0.4

The combined variability is COV = 0.65

Suppose we reduce the variability of these variables to 0.05

The combined variability is now COV = 0.60

If one of the COV's is large, it doesn't do any good to reduce the others, you must reduce the largest one !

Variability Summary

	Source	COV
Stress	Service Loading	0.5
	Environment	0.3
Strength	Materials	0.1
	Manufacturing	0.1
	Surface Finish	0.1

- -

Fatigue Lives	1.0
Analysis Uncertainty	1.0

$$\text{Fatigue life} \propto \left(\frac{\text{Strength}}{\text{Stress}}\right)^5$$

Variability and Uncertainty

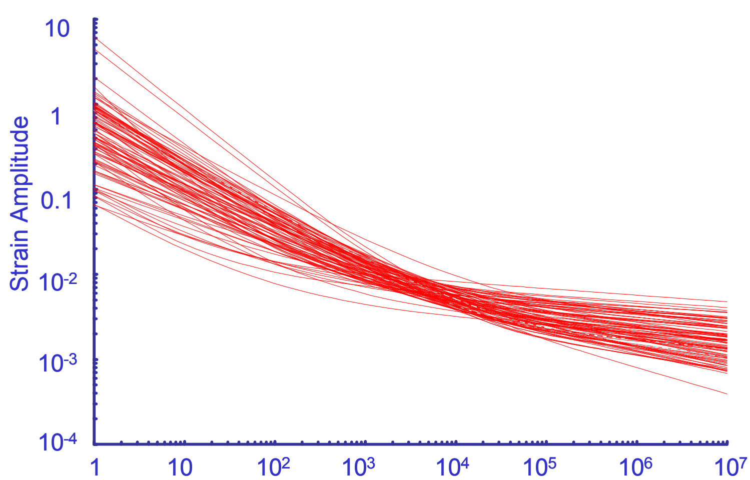
Variability: Every apple on a tree has a different mass.

Uncertainty: The variety of the apple is unknown.

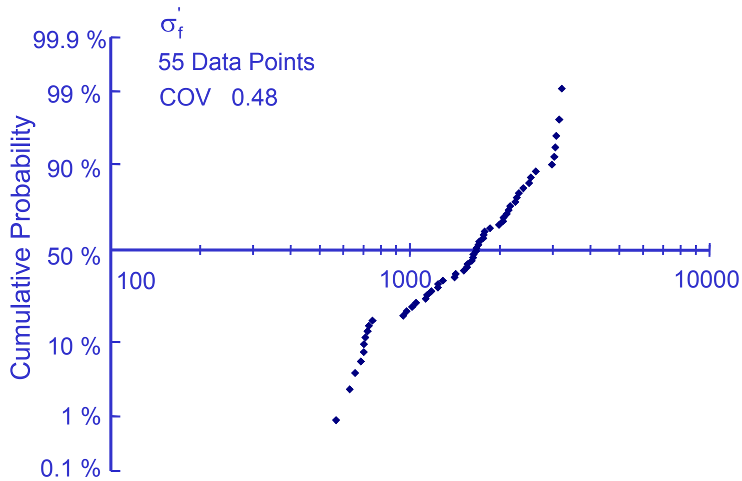
Variability: Multiple samples of the same material

Uncertainty: What is the material

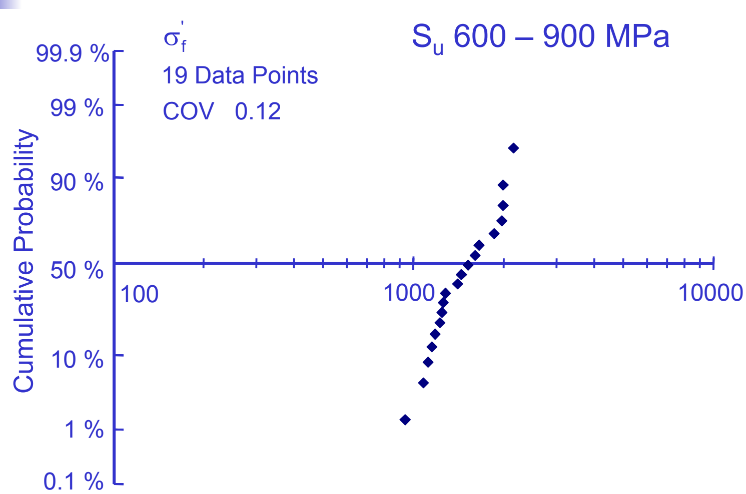
Strain Life Data for 93 Steels



Uncertainty for all Steels



Uncertainty for Structural Steels



Variability and Uncertainty

Fatigue Strength Coefficient

	Variability	Uncertainty	Combined
All Steels	0.12	0.48	0.75
Structural Steel	0.12	0.12	0.24

Quiz

At my last seminar everyone hit a golf ball and we recorded the maximum acceleration.

What is the expected variability ?

Results

	12.9
	7.7
	5.88
	11.1
	15.5
	10.3
	18.1
μ	11.64
σ	4.26
COV	0.37

Probabilistic Aspects of Fatigue

Case Studies

Professor Darrell F. Socie
Department of Mechanical and Industrial Engineering

Probabilistic Aspects of Fatigue

- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- Analysis Methods
- Characterizing Variability
- **Case Studies**
- FatigueCalculator.com
- GlyphWorks

Case Studies

- DARWIN
  - Southwest Research
- Bicycle
- Loading Histories

A Software Framework for Probabilistic Fatigue Life Assessment

ASTM Symposium on
Probabilistic Aspects of Life Prediction
Miami Beach, Florida
November 6-7, 2002

R. C. McClung, M. P. Enright, H. R. Millwater*,
G. R. Leverant, and S. J. Hudak, Jr.
Southwest Research

Slides 6 – 27 used with permission of of Craig McClung

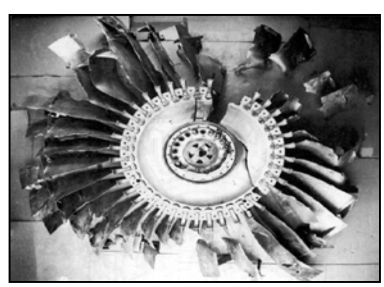
Motivation

UAL Flight 232

July 19,1989

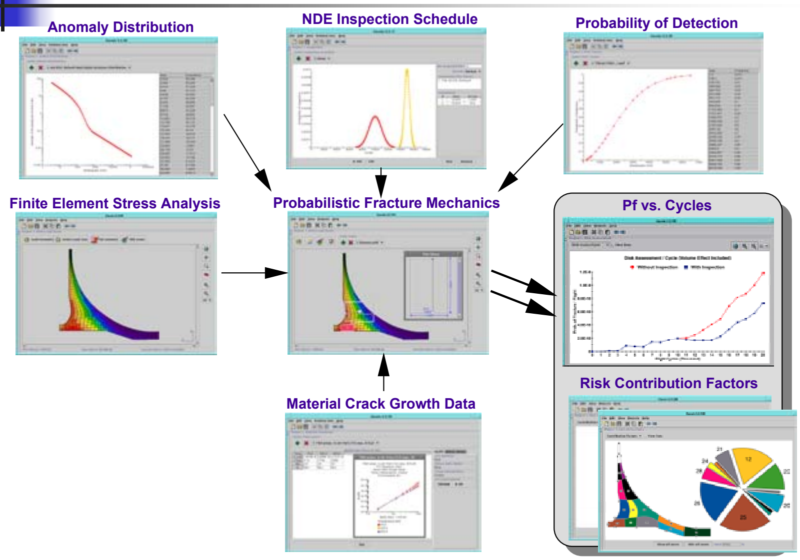
Turbine Disk Failure

Anomalies in titanium engine disks



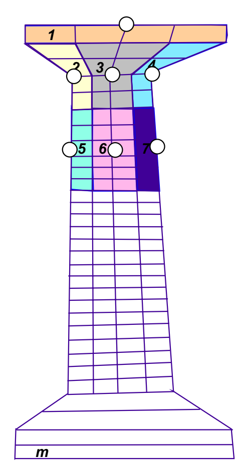
- Hard Alpha
  - Very rare
  - Can cause failure
  - Not addressed by safe life methods
- Enhanced life management process
  - Requested by FAA
  - Developed by engine industry
  - Probabilistic damage tolerance methods
  - Supplement to safe life approach
- SwRI and engine industry developed DARWIN with FAA funding

Probabilistic Damage Tolerance



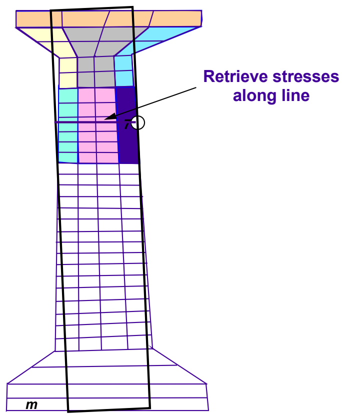
Zone-Based Risk Assessment

- Define zones based on similar stress, inspection, anomaly distribution, lifetime
- Total probability of fracture for zone:
  - (probability of having a defect) x (POF given a defect)
  - Defect probability determined by anomaly distribution, zone volume
  - POF assuming a defect computed with Monte Carlo sampling or advanced methods
- POF for disk obtained by summing zone probabilities
- As individual zones become smaller (number of zones increases), risk converges down to “exact” answer

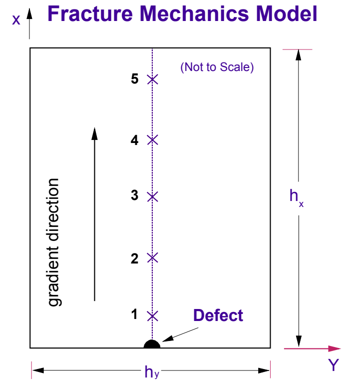


Fracture Mechanics Model of Zone

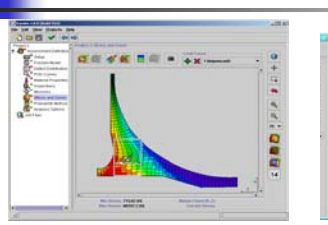
Finite Element Model



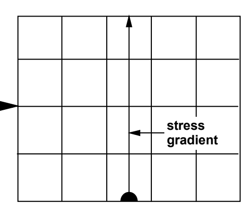
Fracture Mechanics Model



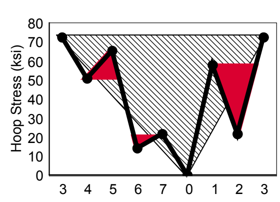
Stress Processing



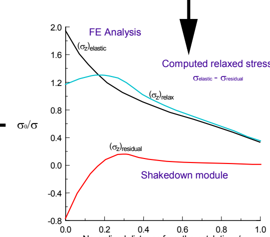
FE Stresses and plate definition



Stress gradient extraction



Rainflow stress pairing

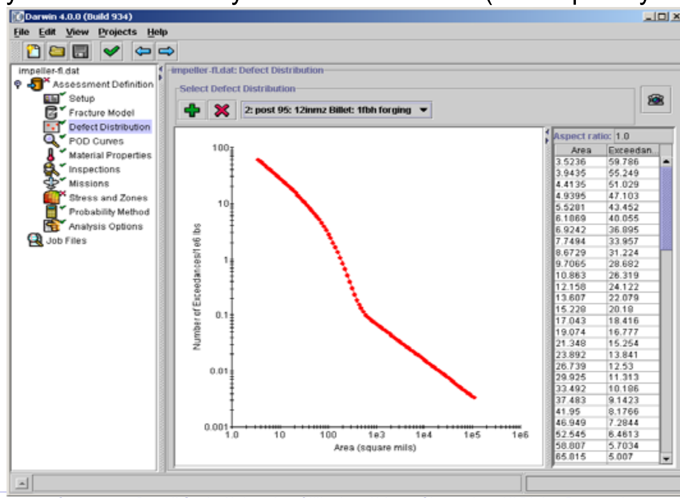


Residual stress analysis

Anomaly Distribution

of anomalies per volume of material as function of defect size

Library of default anomaly distributions for HA (developed by RISC)

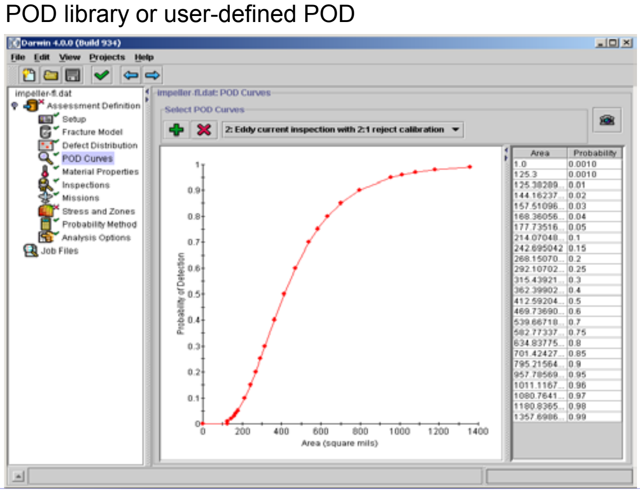


Probability of Detection Curves

Define probability of NDE flaw detection as function of flaw size

Can specify different PODs for different zones, schedules

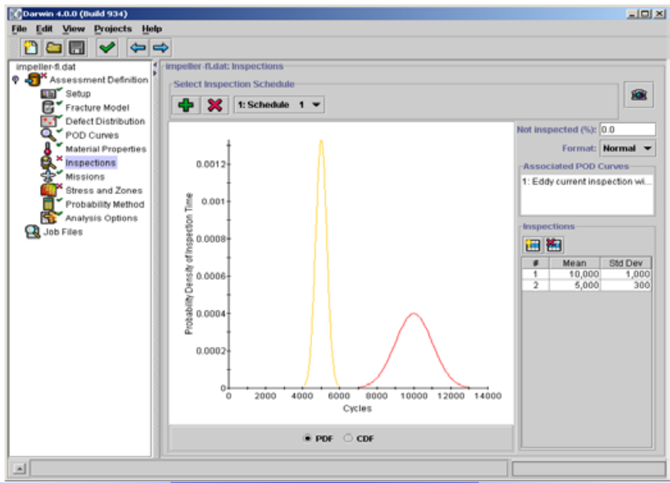
Built-in POD library or user-defined POD



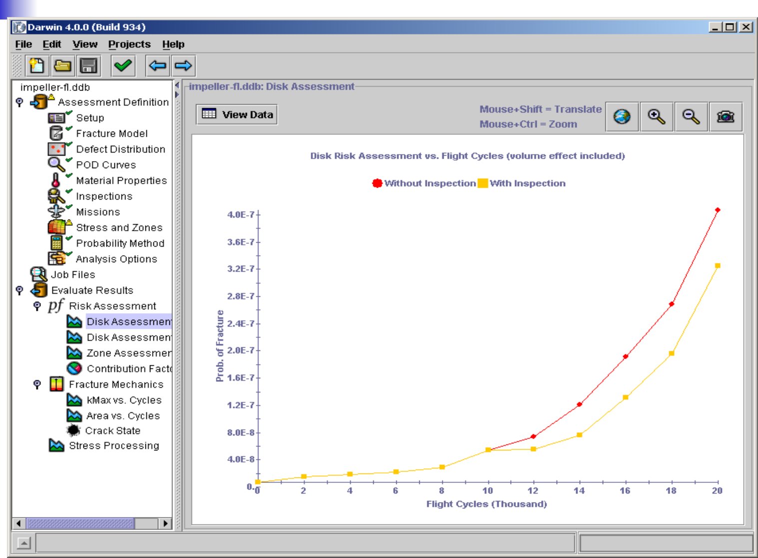
Random Inspection Time

“Opportunity Inspections” during on-condition maintenance

Inspection time modeled with Normal distribution or CDF table

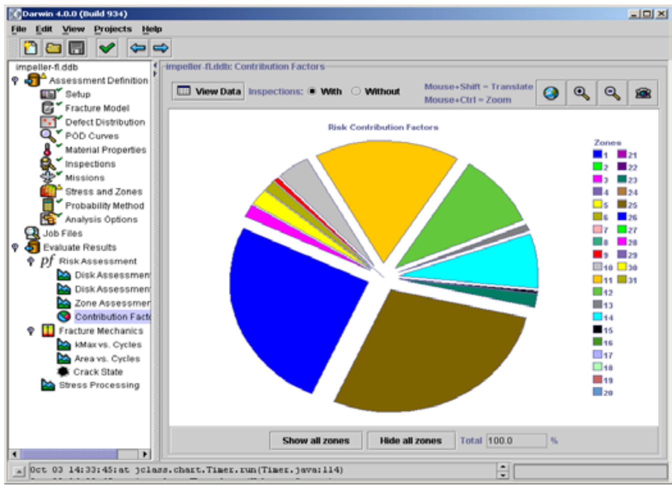


Output: Risk vs. Flight Cycles



Output: Risk Contribution Factors

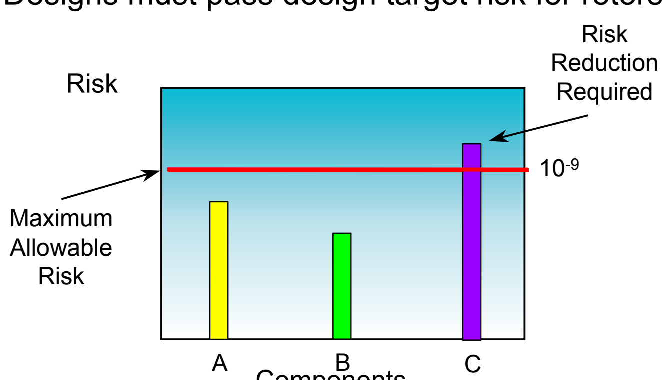
Identify regions of component with highest risk



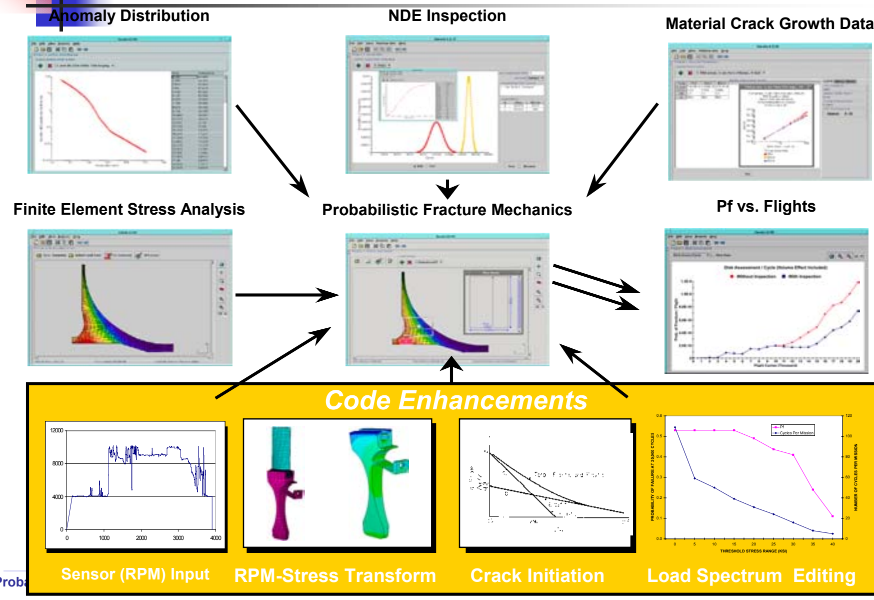
Implementation in Industry

FAA Advisory Circular 33.14 requests risk assessment be performed for all new titanium rotor designs

Designs must pass design target risk for rotors



DARWIN for Prognosis Studies

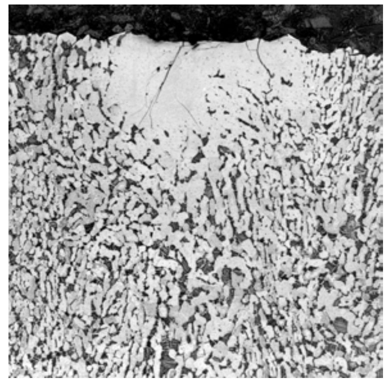


Three Sources of Variability

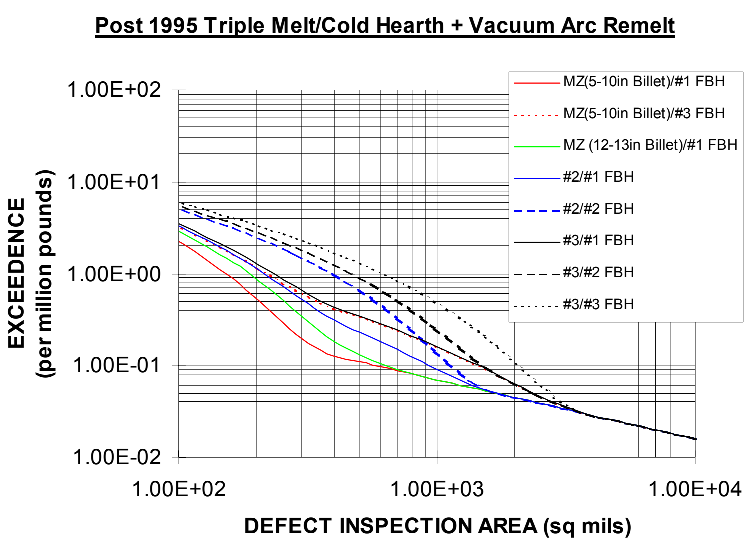
- Anomaly size (initial crack size)
- FCG properties (life scatter)
- Mission histories (stress scatter)

Hard Alpha Defects in Titanium

- Initial DARWIN focus on Hard Alpha
  - Small brittle zone in microstructure
  - Alpha phase stabilized by N accidentally introduced during melting
  - Cracks initiate quickly
- Extensive industry effort to develop HA distribution

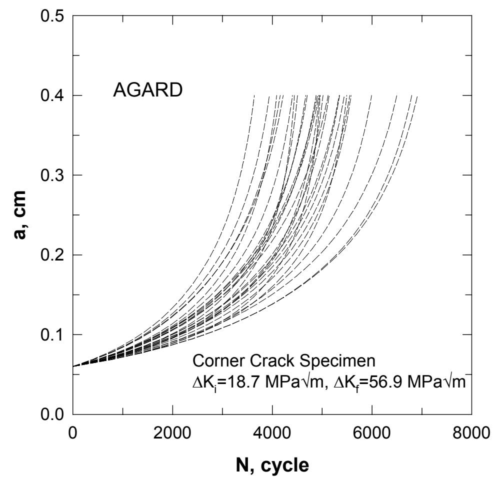


Resulting Anomaly Distributions



FCG Simulations for AGARD Data

- Use individual fits to generate set of *a* vs. *N* curves for identical conditions
- Characterize resulting scatter in total propagation life
- Lognormal distribution appropriate in most cases



Engine Usage Variability

Stress/Speed:

$\Delta\sigma \propto (RPM)^2$

Total Cyclic Life (LCF):

$N_f = N_i + N_p$

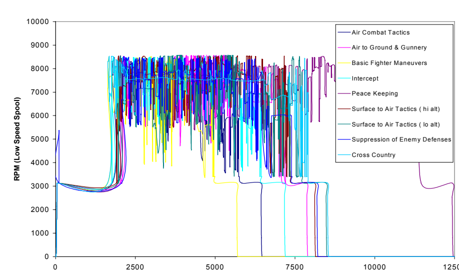
$N_i \propto \Delta\sigma^{3\text{-}5}$

$N_p \propto \Delta\sigma^{3\text{-}4}$

Life/Speed:

$N_f \propto (RPM)^6$

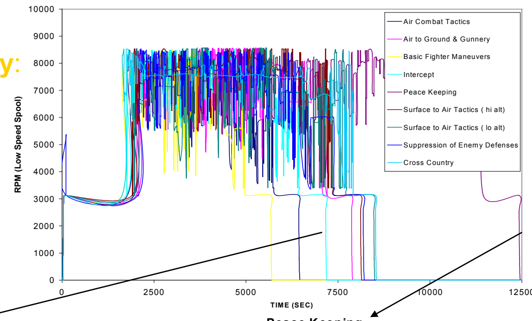
Component life is very sensitive to actual usage



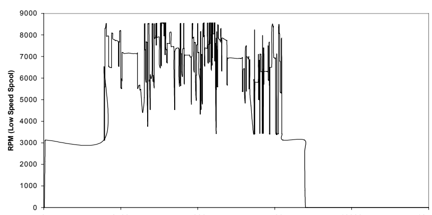
Usage Variability

Components of Usage Variability:

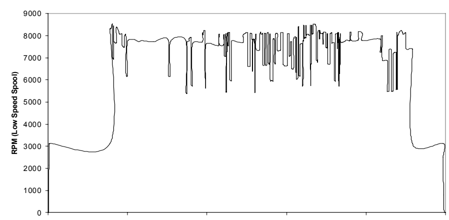
- Mission type
- Mission-to-mission variability
- Mission mixing variability



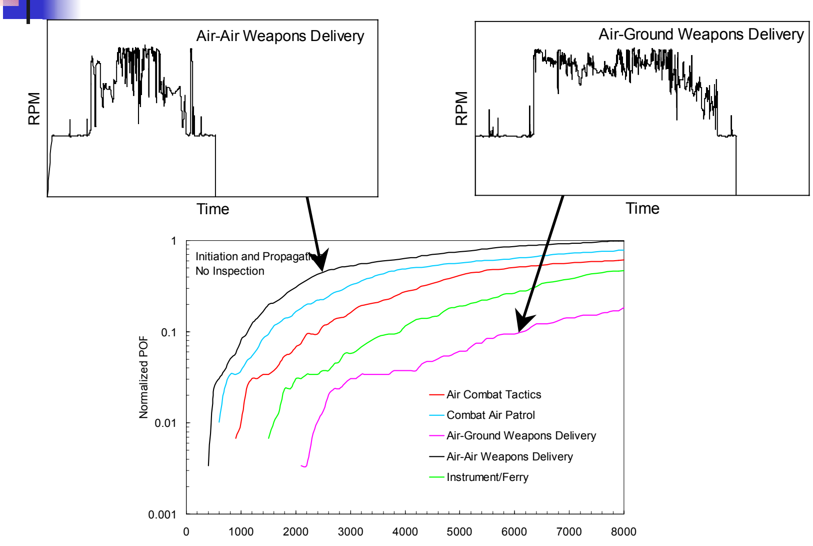
Surface to Air Tactics (lo alt)



Peace Keeping



Variability in Mission Type



Web Site: www.darwin.swri.org

Fatigue Design and Reliability in the Automotive Industry

Fatigue Design and Reliability
ESIS Publication 23

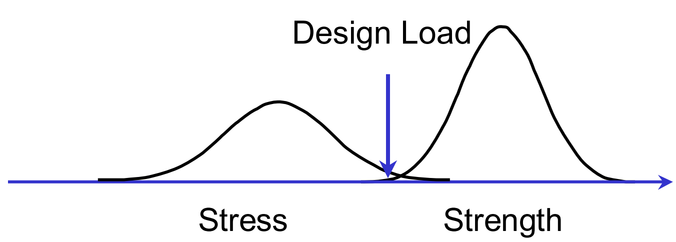
J-J. Thomas, G. Perroud, A. Bignonnet and D. Monnet

PSA Peugeot Citroën

Fatigue Design at PSA

Fatigue design at PSA is done with a probabilistic approach that includes analysis of customer usage, production scatter, definition of the appropriate design loads and an acceptance testing criterion.

Stress-Strength Interference



Reliability = 1 – P(Stress > Strength)

How do you really get these distributions ?

How do you establish a design load ?

How do you validate the design ?

Stress Distribution

Variability in loading has two components,
how it is used and how it is driven.

Car Usage
Highway, city, fully loaded, empty etc.

Driving Style
passive, aggressive etc.

The usage of a car is independent of the owners driving style so that the distributions of car usage and driving style can be obtained separately.

Car Usage

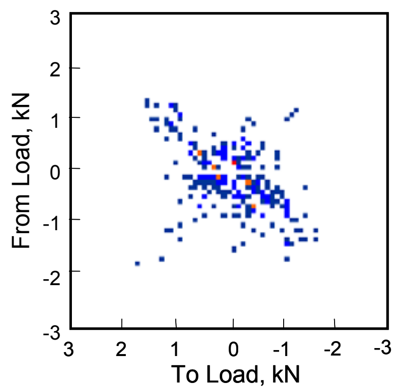
Customer surveys

k	l			c_k %	r_{kl} %
1		Unloaded		27	
	1		Highway		10
	2		Good Road		25
	3		Mountain		40
	4		City		25
2		Half Load		58	
	1		Highway		5
	2		Good Road		30
	3		Mountain		30
	4		City		35
3		Fully Loaded		15	
	1		Highway		15
	2		Good Road		25
	3		Mountain		40
	4		City		20

12 Customer Usage Categories

Owner Behavior

Instrumented Vehicles

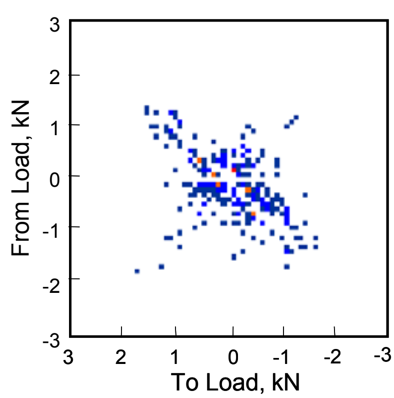


Extensive field testing for each customer usage category produces a large number of histograms.

Let the usage histogram be denoted U_{kl}

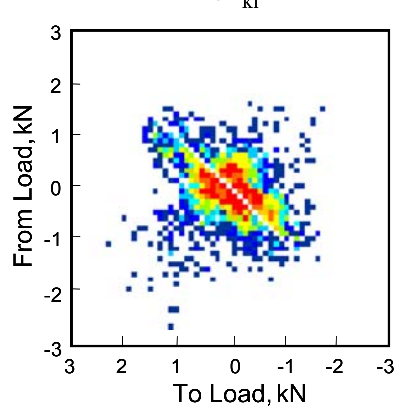
Extrapolation

U_{kl}



Measured data, e.g. 1,000 km

U^*_{kl}



Extrapolated data, 300,000 km

Virtual Customers

Thousands of virtual customers can now be generated by combining customer usage with driving style.

$$[U^i] = N \sum c_k \, r_{kl} \, [U^*_{kl}]$$

$[U^i]$ rainflow histogram for an individual car

N kilometers

c_k car loading, %

r_{kl} car usage, %

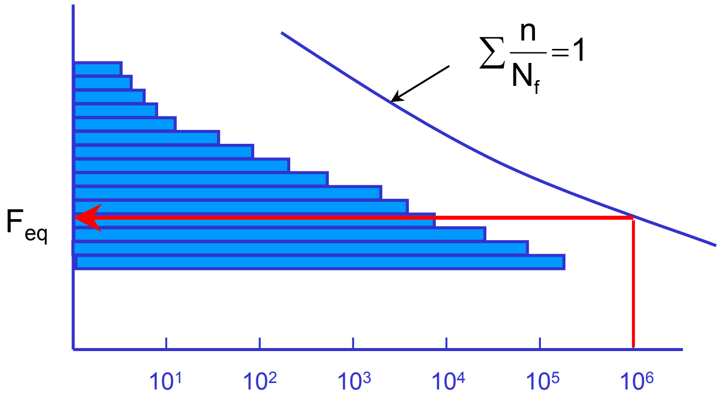
$[U^*_{kl}]$ distributions of rainflow histograms for different car usage classifications

Design Loads

Initial design is done on the basis of a single constant amplitude load, F_{eq}

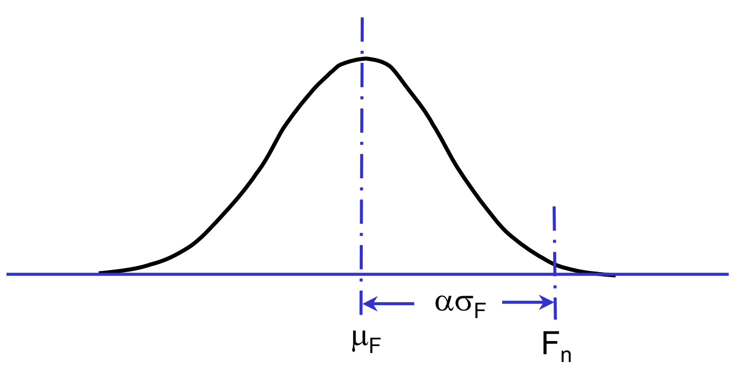
Find a constant amplitude load and number of cycles that will produce the same fatigue damage as the customer operating a car for the design life.

Equivalent Fatigue Loading



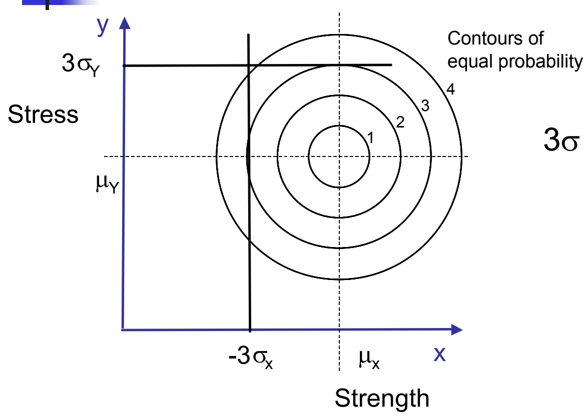
$$F_{eq} = \left(\frac{\sum \Delta F^m}{10^6} \right)^{\frac{1}{m}}$$

Distribution of F_{eq}



Design Customer: $F_n = \mu_F + \alpha\sigma_F$

Choosing α

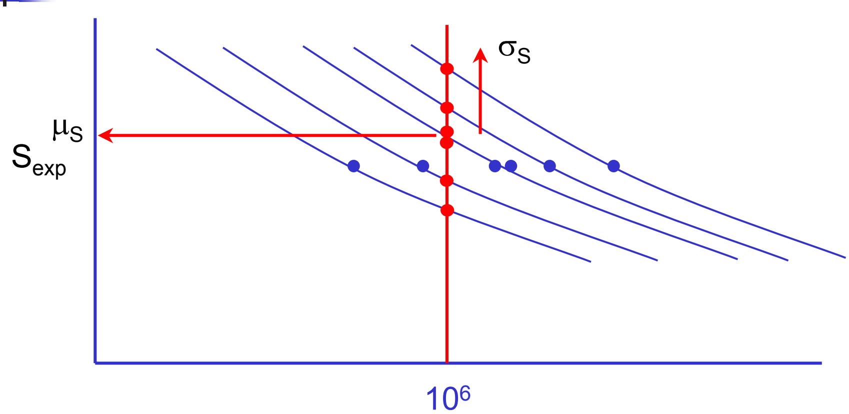


$3\sigma \;\; P(s < S) = 2.3 \; 10^{-3}$

If we use 3σ on both stress and strength

$$P(\text{failure}) = P(\Sigma \geq s \cap s \leq S) = 5.3 \;\; 10^{-6} \approx 4.5\sigma$$

Distribution of Strength



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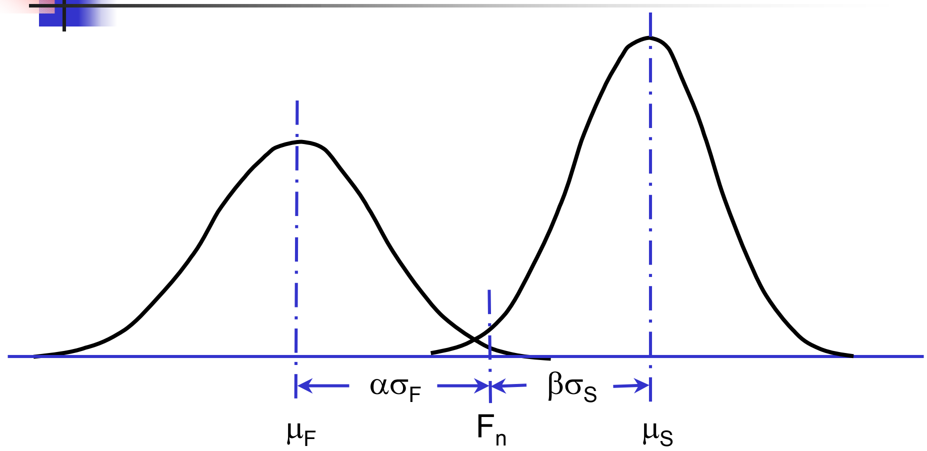
What Strength is Needed?



Some Statistics

$$Z = S - F$$

Suppose we want a probability of failure of 1 in 50,000

$$P_f = 2 \times 10^{-5}$$

$$\Phi^{-1}(P_f) = 4.1 = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_S - \mu_F}{\sqrt{\sigma_S^2 + \sigma_F^2}}$$

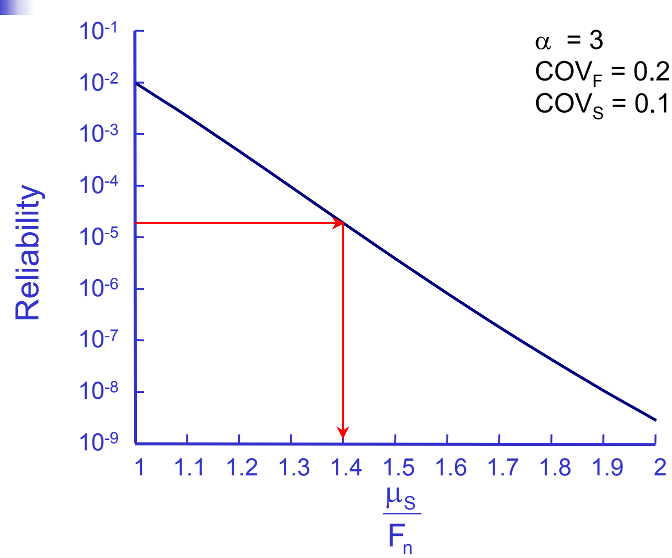
$$F_n = \mu_S(\ 1 - \beta\ COV_S\ )$$
$$F_n = \mu_F(\ 1 + \alpha\ COV_F\ )$$

Mean Component Strength

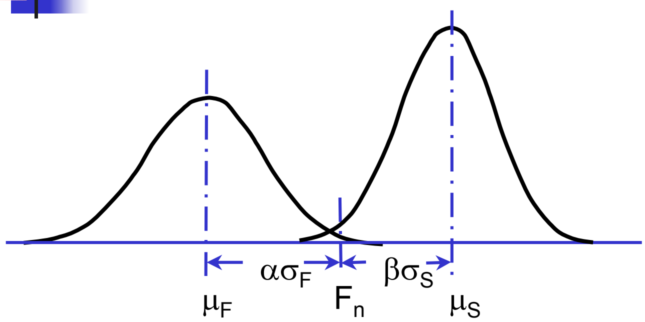
$$\Phi^{-1}(P_f) = 4.1 = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_S - \mu_F}{\sqrt{\sigma_S^2 + \sigma_F^2}}$$

$$\Phi^{-1}(P_f) = \frac{\dfrac{\mu_S}{F_n} - \dfrac{1}{1+\alpha\, COV_F}}{\sqrt{\left(\dfrac{\mu_S}{F_n} COV_S\right)^2 + \left(\dfrac{COV_F}{1+\alpha\, COV_F}\right)^2}}$$

Reliability



Mean Strength



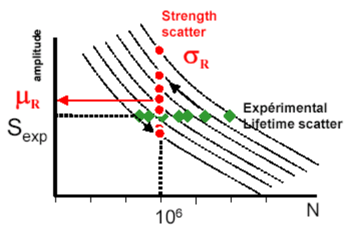
μ_S = 1.4 F_n for a reliability of 2 x 10^{-5}

$$\beta = \frac{1 - \frac{F_n}{\mu_S}}{COV_S} = 2.85$$

Design

- Design calculations are made with loads F_n
  - Large database relates F_n to vehicle parameters so that a new vehicle can be designed from historical measurements
- Fatigue calculations are made with material properties β standard deviations from the mean

Component Testing



Component strength not life !

Test load is frequently higher than F_n to get failures near the design life

Interpreting the Test Data

Component tests are done with small sample sizes

Confidence limits

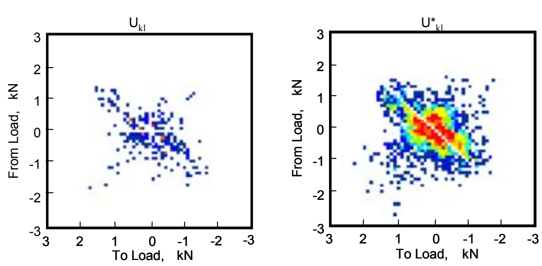
$$\frac{(N-1)s_x^2}{\chi^2_{1-\alpha,N-1}} \leq \sigma_x^2$$

$$\Phi^{-1}(P_f) = \frac{\dfrac{\mu_S}{F_n} - \dfrac{1}{1+\alpha\,COV_F}}{\sqrt{\left(\dfrac{\mu_S}{F_n}COV_S\right)^2 \dfrac{N-1}{\chi^2_{1-\alpha,N-1}} + \left(\dfrac{COV_F}{1+\alpha\,COV_F}\right)^2}}$$

Validation

Full scale vehicle simulation done at the end for design final validation

Extrapolation

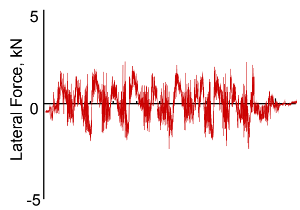


Measured data, e.g. 1,000 km

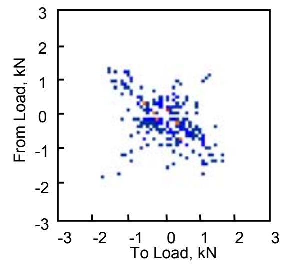
Extrapolated data, 300,000 km

How does this process work?

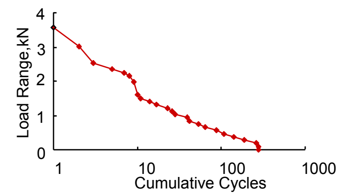
Service Loading Spectra



Time history



Rainflow Histogram



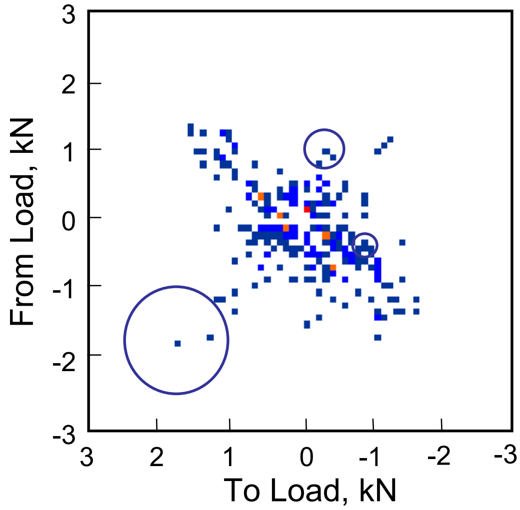
Exceedance Diagram

Problem Statement

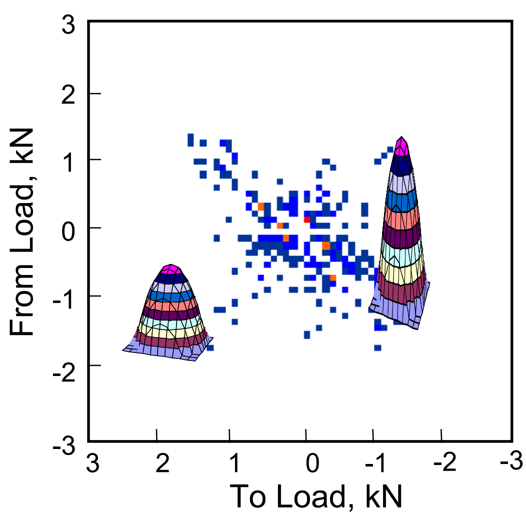
Given a rainflow histogram for a single user, extrapolate to longer times

Given rainflow histograms for multiple users, extrapolate to more users

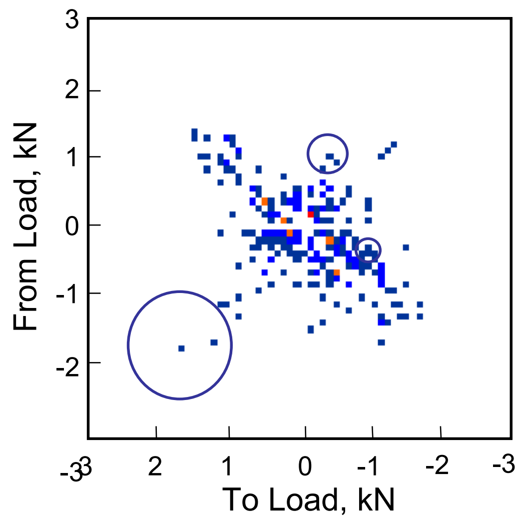
Probability Density



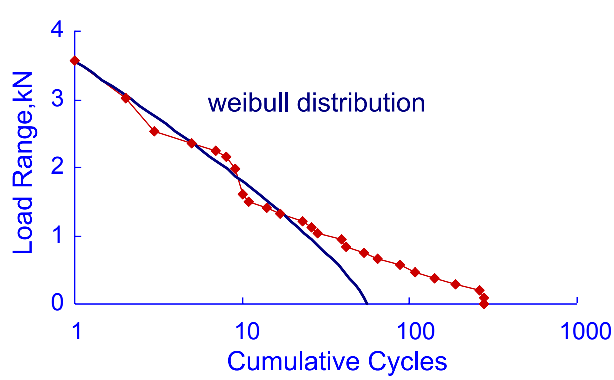
Kernel Smoothing



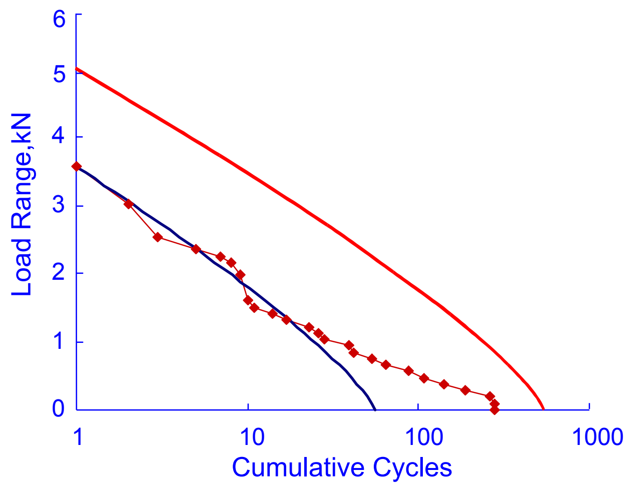
Sparse Data



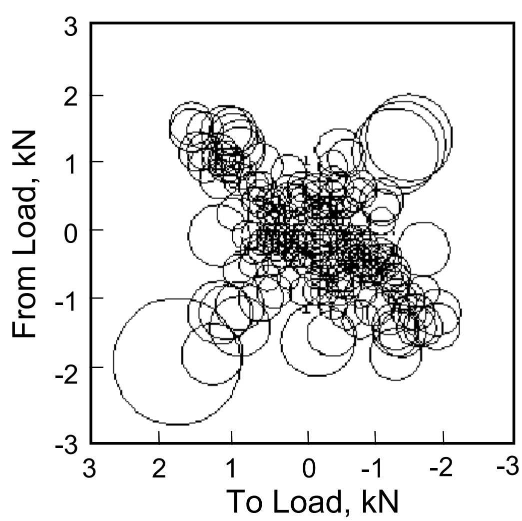
Exceedance Plot of 1 Lap



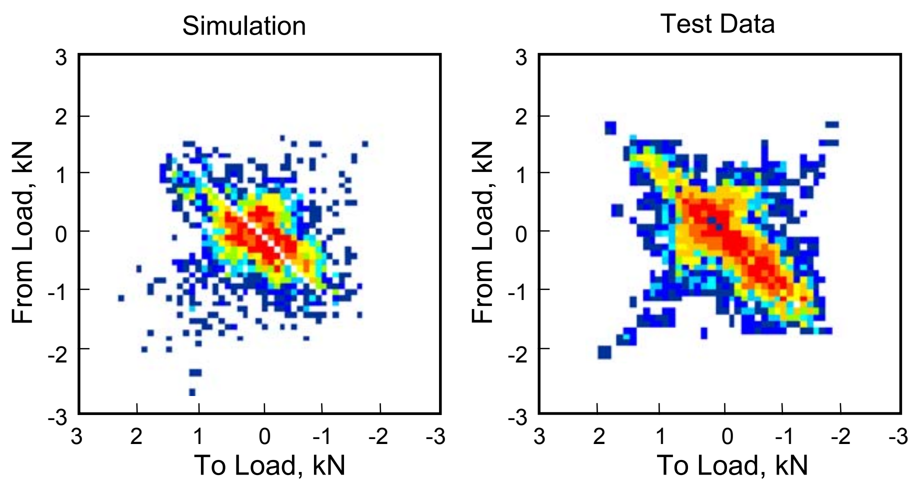
10X Extrapolation



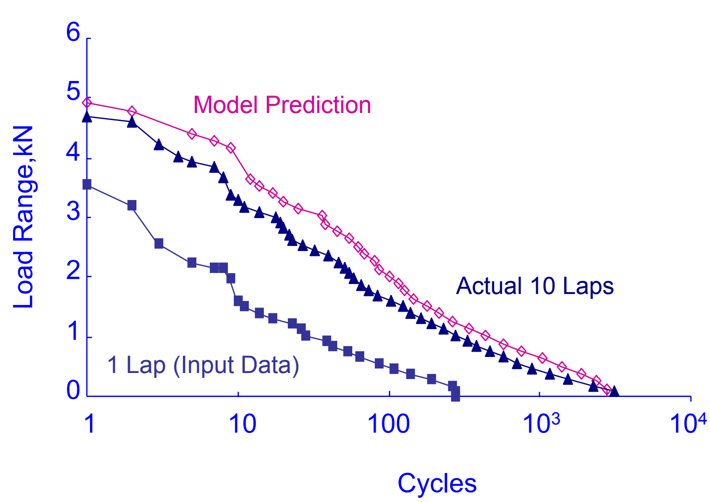
Probability Density



Results



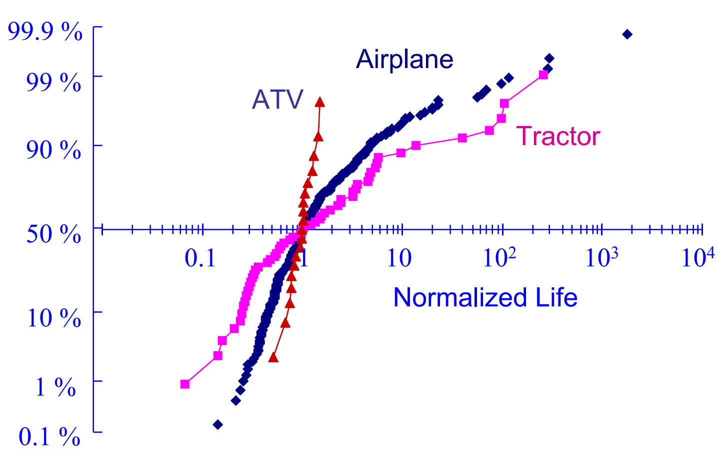
Exceedance Diagram



Problem Statement

- Given a rainflow histogram for a single user, extrapolate to longer times
- Given rainflow histograms for multiple users, extropolate to more users

Extrapolated Data Sets

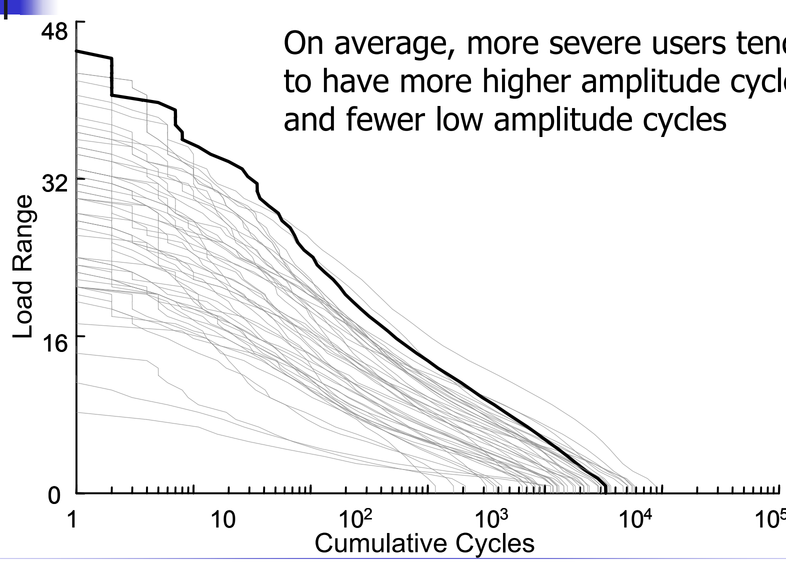


Issues

In the first problem the number of cycles is known but the variability is unknown and must be estimated

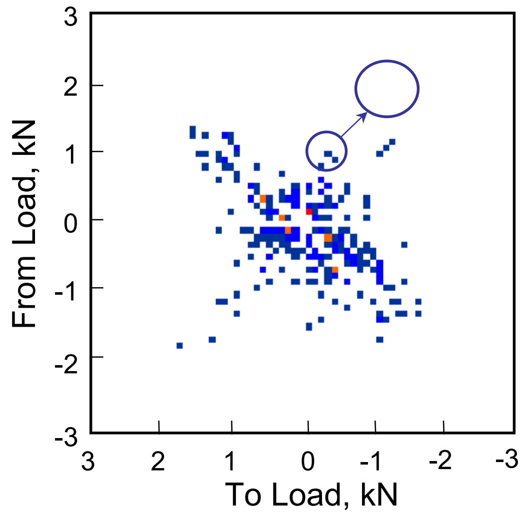
In the second problem the variability is known but the number and location of cycles is unknown and must be estimated

Assumption

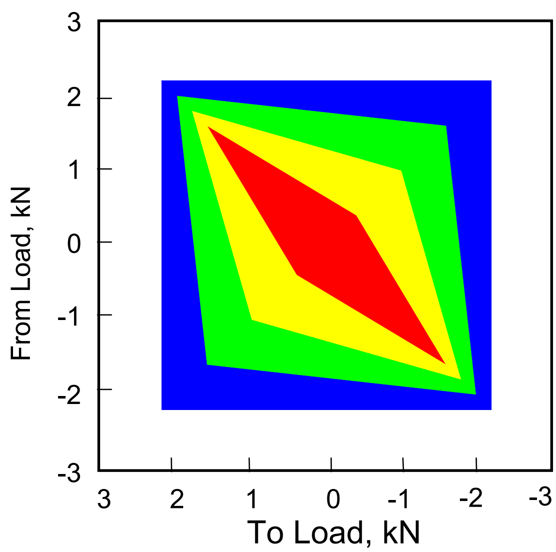


On average, more severe users tend to have more higher amplitude cycles and fewer low amplitude cycles

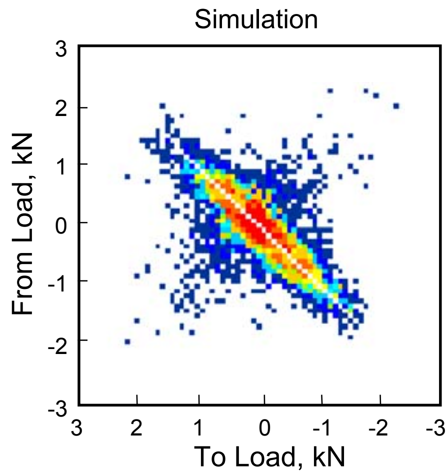
Translation

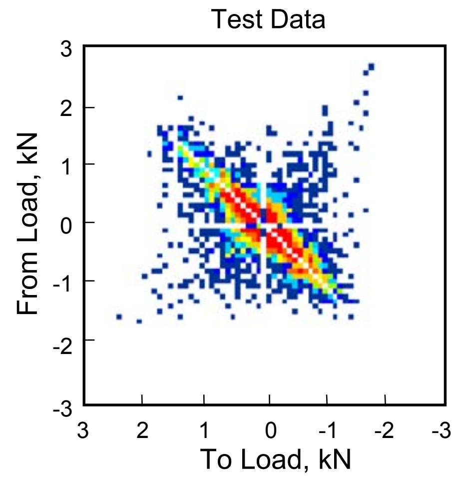


Damage Regions

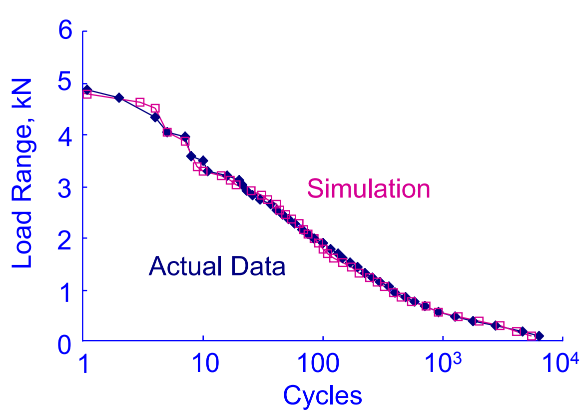


ATV Data - Most Damaging in 19

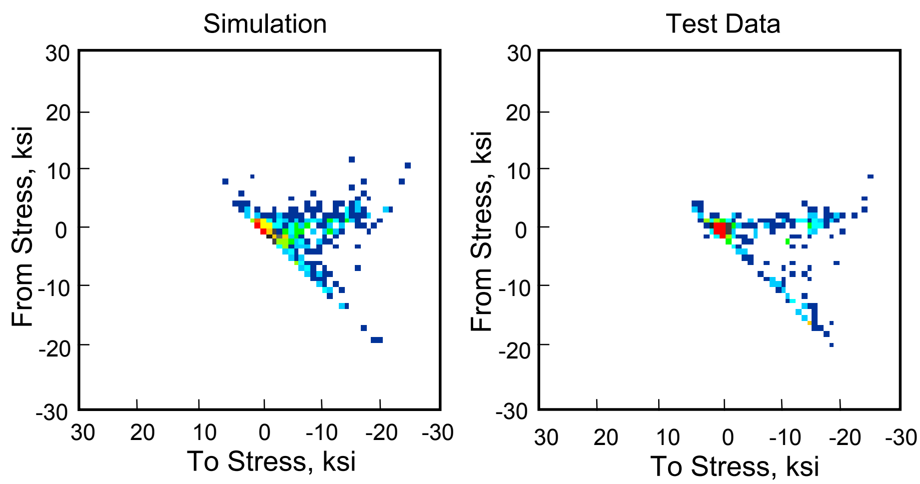




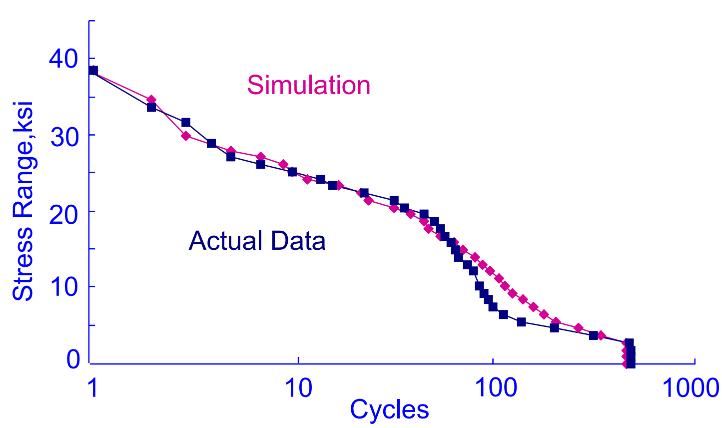
ATV Exceedance



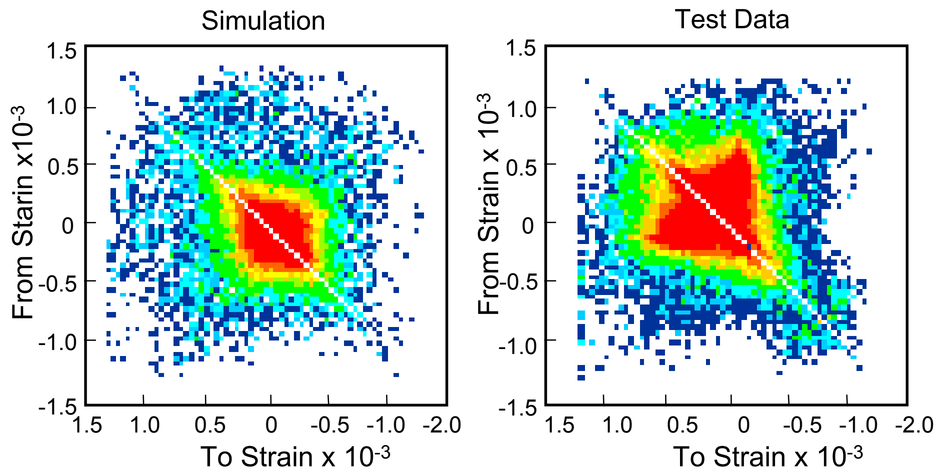
Airplane Data - Most Damaging in 334



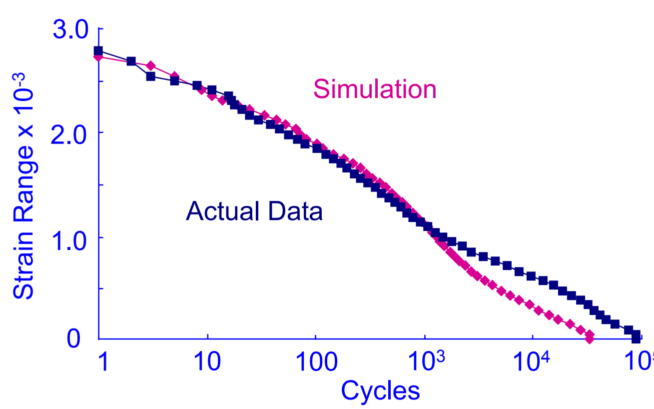
Airplane Exceedance



Tractor Data - Most Damaging in 54



Tractor Exceedance



Probabilistic Aspects of Fatigue

www.FatigueCalculator.com

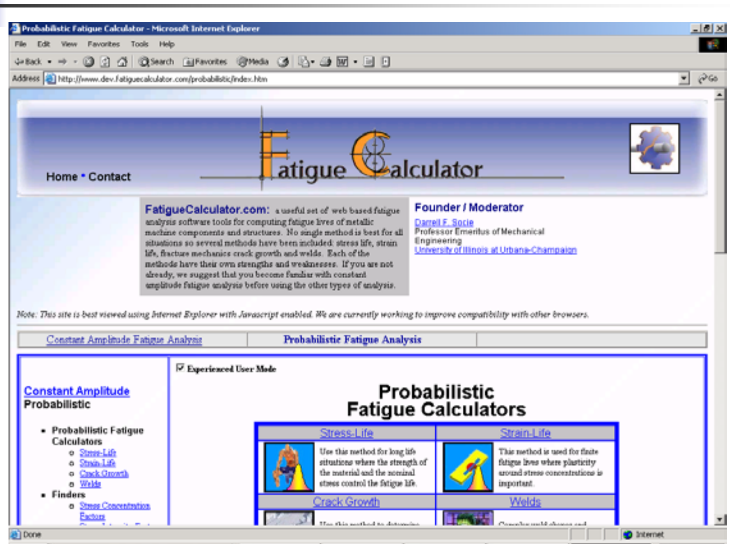
Professor Darrell F. Socie
Department of Mechanical and Industrial Engineering

Probabilistic Aspects of Fatigue

- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- Analysis Methods
- Characterizing Variability
- Case Studies
- **FatigueCalculator.com**
- GlyphWorks

www.FatigueCalculator.com

Probabilistic Fatigue Analysis

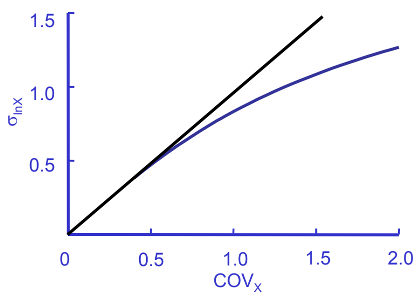


Modeling Variability

Central Limit Theorem:

Products: $Z = X_1 \bullet X_2 \bullet X_3 \bullet X_4 \bullet \ldots.. X_n$

$Z \rightarrow$ LogNormal as n increases



$\sigma_{lnX} \sim COV_X$

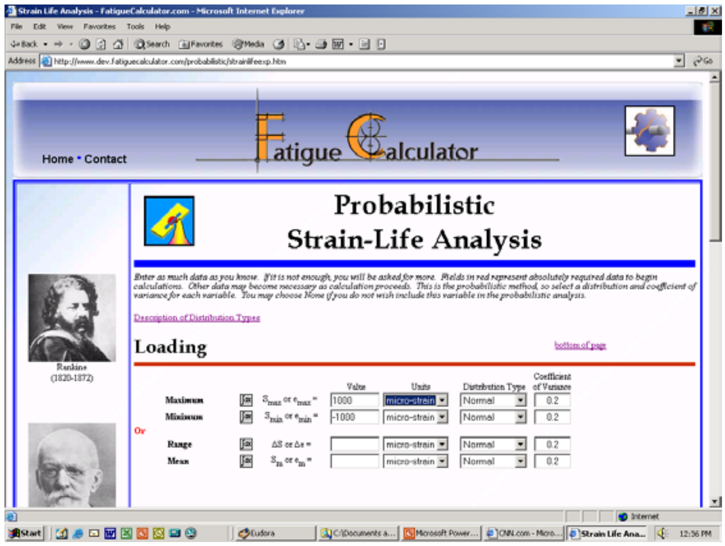
COV_X is a good measure of variability

COV and LogNormal Distributions

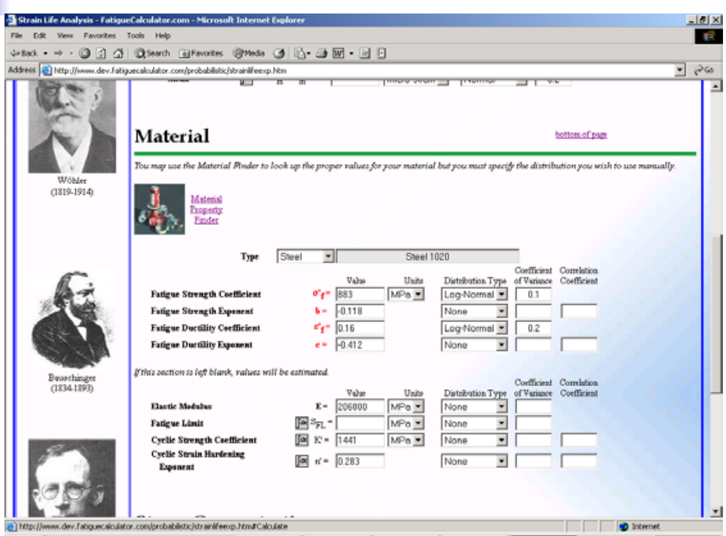
COV_X	Standard Deviation, lnx		
	1 68.3%	2 95.4%	3 99.7%
0.05	1.05	1.11	1.16
0.1	1.10	1.23	1.33
0.25	1.28	1.66	2.04
0.5	1.60	2.64	3.92
1	2.30	5.53	11.1

99.7% of the data is within a factor of ± 1.33 of the mean for a COV = 0.1

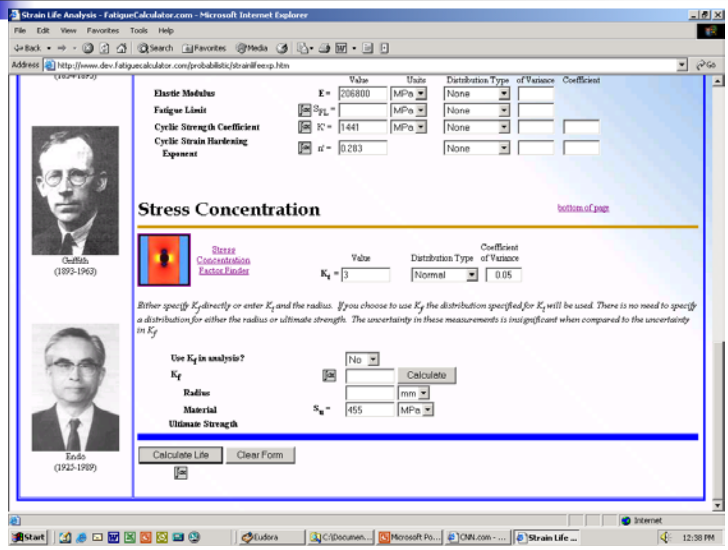
Strain Life Analysis



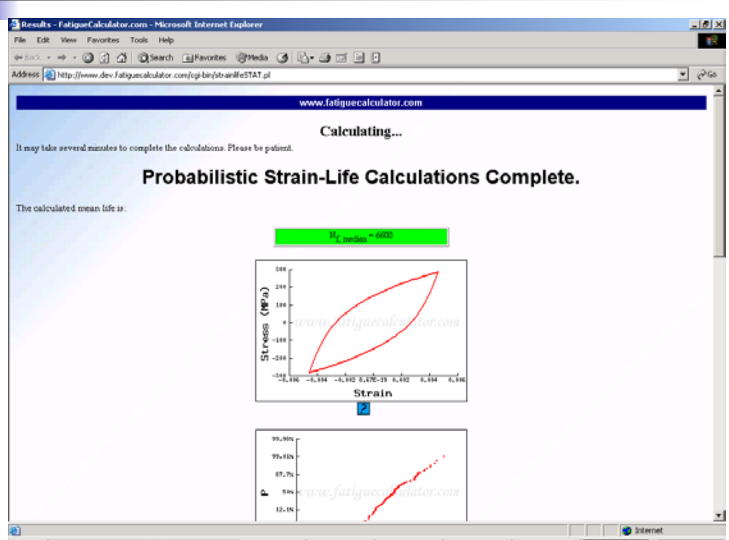
Material Properties



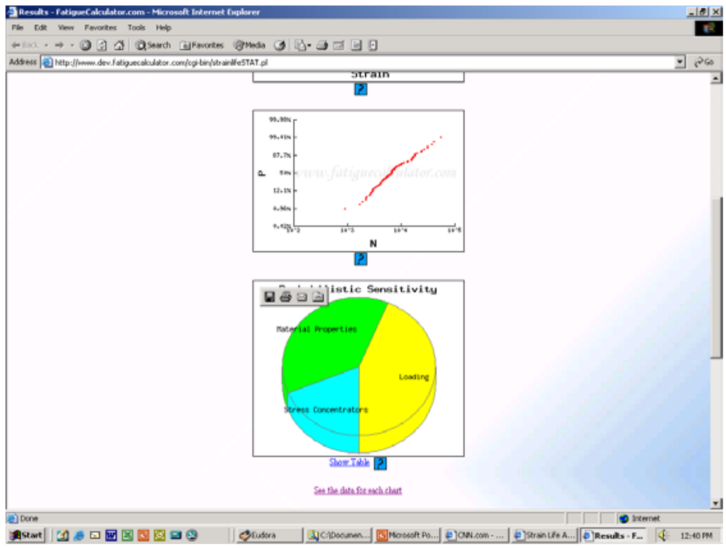
Stress Concentration



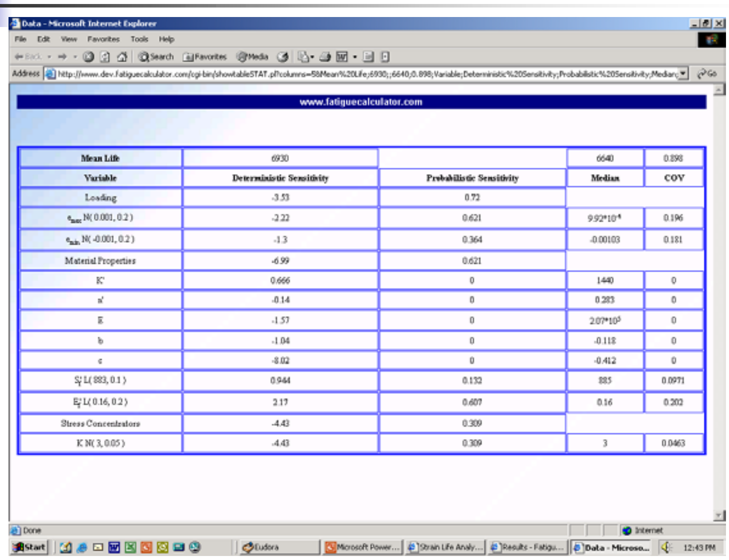
Results



Results (continued)



Results (continued)



www.fatiguecalculator.com

Mean Life	6930		6640	0.898
Variable	**Deterministic Sensitivity**	**Probabilistic Sensitivity**	**Median**	**COV**
Loading	-3.53	0.72		
e_{max} N(0.001, 0.2)	-2.22	0.621	$9.92*10^{-4}$	0.196
e_{min} N(-0.001, 0.2)	-1.3	0.364	-0.00103	0.181
Material Properties	-6.99	0.621		
K'	0.666	0	1440	0
n'	-0.14	0	0.283	0
E	-1.57	0	$2.07*10^{5}$	0
b	-1.04	0	-0.118	0
c	-8.02	0	-0.412	0
S_f' L(883, 0.1)	0.944	0.132	885	0.0971
E_f' L(0.16, 0.2)	2.17	0.607	0.16	0.202
Stress Concentrators	-4.43	0.309		
K N(3, 0.05)	-4.43	0.309	3	0.0463

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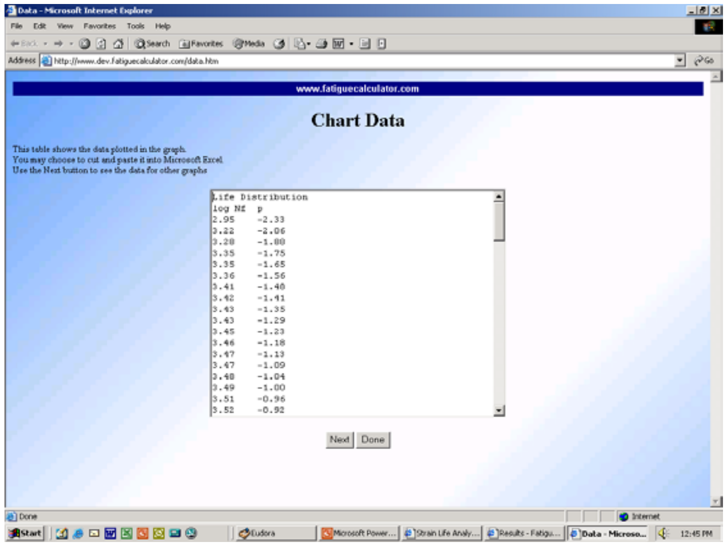
Results (continued)



Ten Simulations

	Life	COV
	6470	0.959
	6930	0.898
	6710	0.688
	6640	0.908
	6580	0.869
	6470	0.959
	7010	0.723
	6690	0.908
	6170	0.791
	6560	0.971
Mean	6623	0.8674
COV	0.038	0.114

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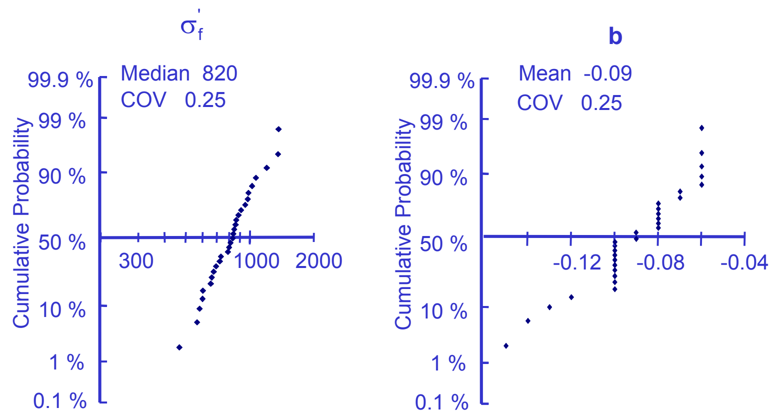
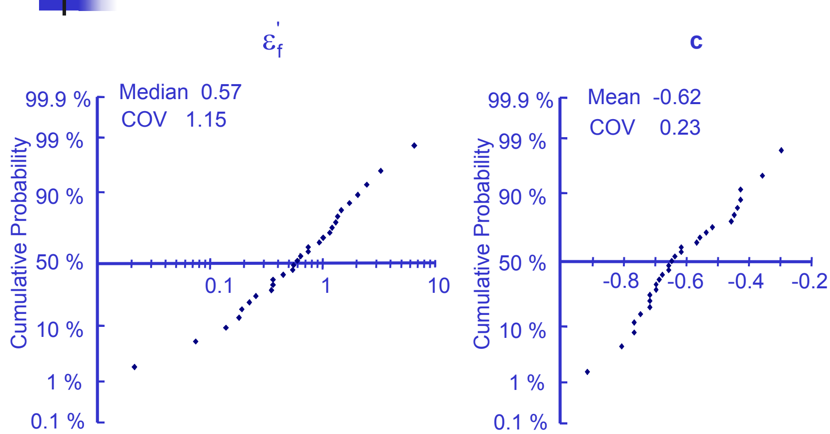
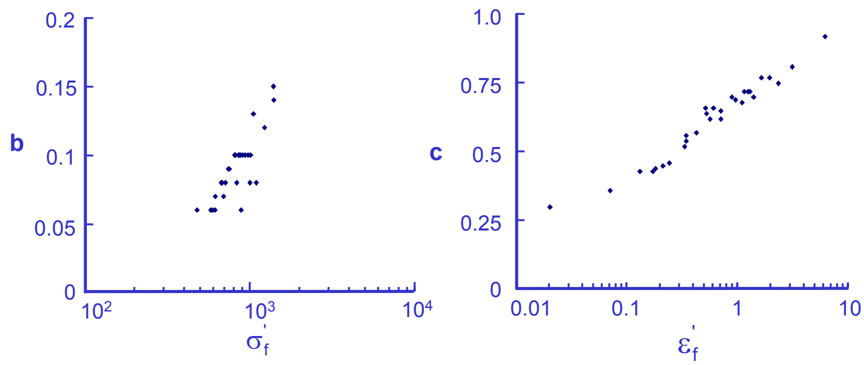
29 Individual Data Sets



29 Individual Data Sets (continued)

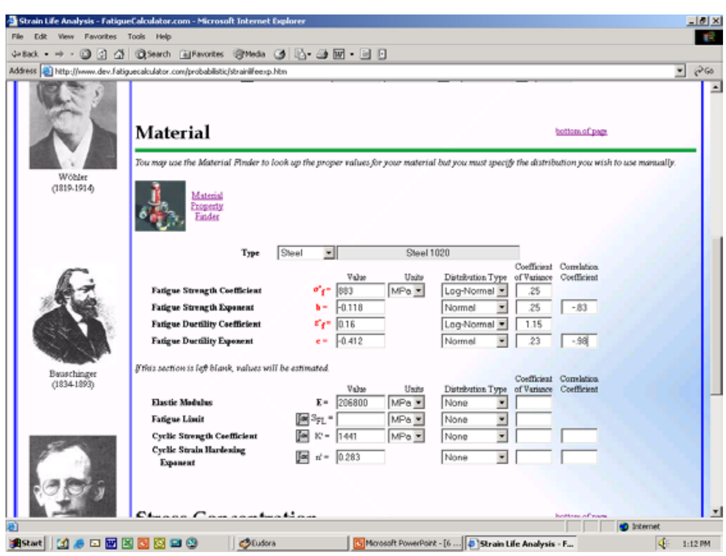


Correlation

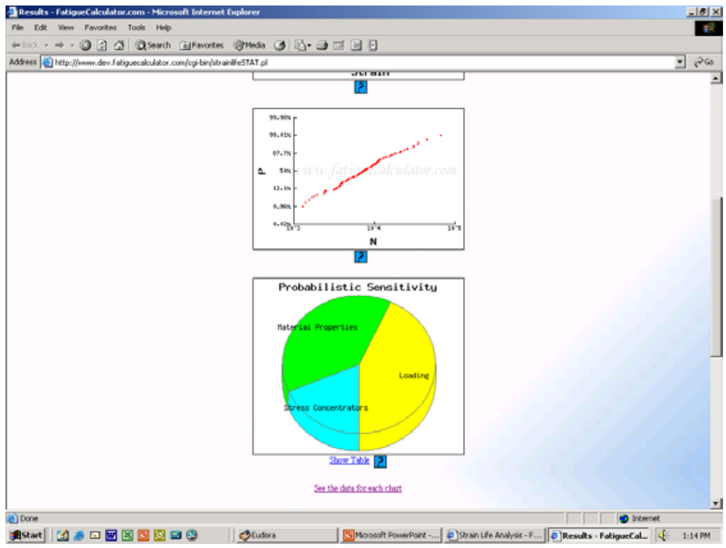


ρ = -0.828 ρ = -0.976

Correlated Variables



Results (continued)



Results (continued)

www.fatiguecalculator.com

Mean Life	7440		6640	0.913
Variable	**Deterministic Sensitivity**	**Probabilistic Sensitivity**	**Median**	**COV**
Loading	-3.53	0.71		
e_{max} N(0.001, 0.2)	-2.22	0.613	$9.94*10^{-4}$	0.202
e_{min} N(-0.001, 0.2)	-1.3	0.359	-0.00101	0.219
Material Properties	-6.99	0.635		
K'	0.666	0	1440	0
n'	-0.14	0	0.283	0
E	-1.57	0	$2.07*10^{5}$	0
b N(-0.118, .25), CC=-.83	-1.04	0.2	-0.119	0.24
c N(-0.412, .23)	-8.02	0.506	-0.41	0.0487
S_f' L(883, .25)	0.944	0.325	861	0.262
E_f' L(0.16, 1.15)	2.17	0.0344	0.16	0.0106
Stress Concentrators	-4.43	0.305		
K N(3, 0.05)	-4.43	0.305	3	0.0472

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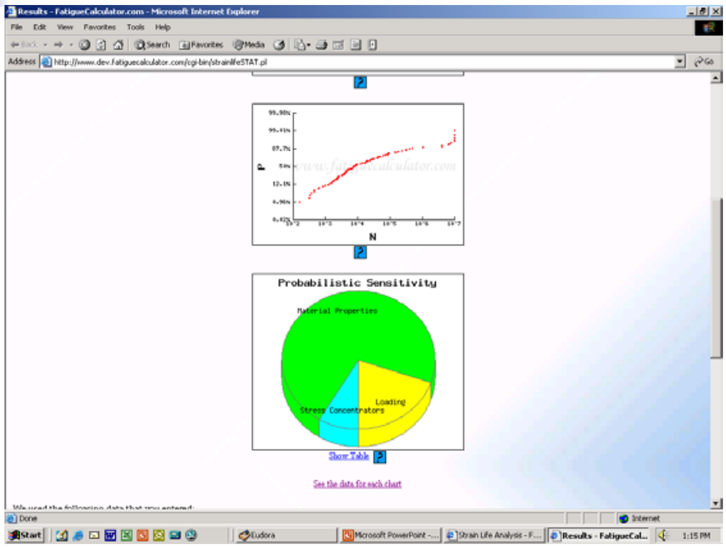
Uncorrelated Variables



Results (continued)

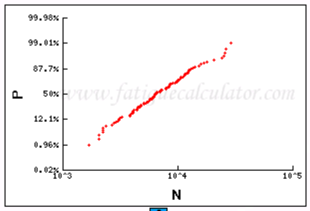
Data - Microsoft Internet Explorer

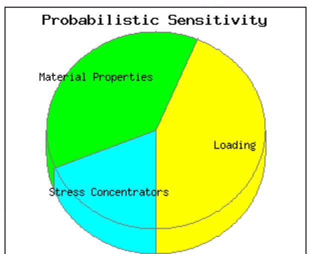
www.fatiguecalculator.com

Mean Life	$1.900*10^4$		6640	155
Variable	**Deterministic Sensitivity**	**Probabilistic Sensitivity**	**Median**	**COV**
Loading	-3.53	0.263		
e_{max} N(0.001, 0.2)	-2.22	0.227	$9.94*10^{-4}$	0.202
e_{min} N(-0.001, 0.2)	-1.3	0.133	-0.00101	0.219
Material Properties	-6.99	0.958		
K'	0.666	0	1440	0
n'	-0.14	0	0.283	0
E	-1.57	0	$2.07*10^5$	0
b N(-0.118, .25)	-1.04	0.132	-0.119	0.26
c N(-0.412, .23)	-8.02	0.941	-0.401	0.25
S_f' L(883, .25)	0.944	0.12	861	0.262
E_f' L(0.16, 1.15)	2.17	0.0127	0.16	0.0106
Stress Concentrators	-4.43	0.113		
K N(3, 0.05)	-4.43	0.113	3	0.0472

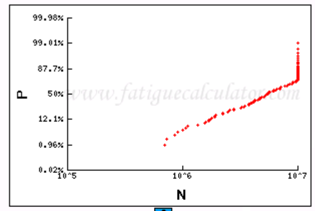
Strain Amplitude

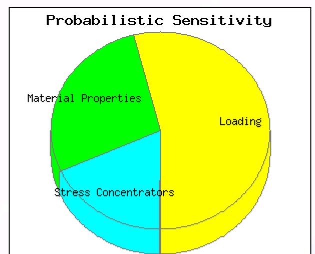
± 1000



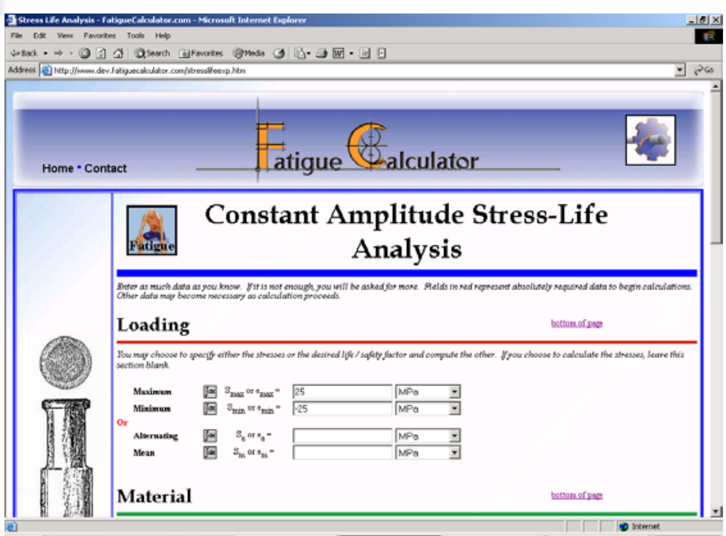


± 250

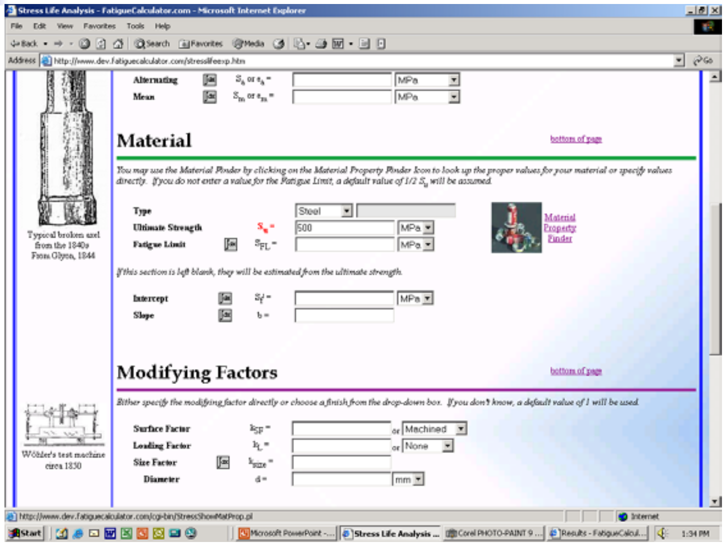




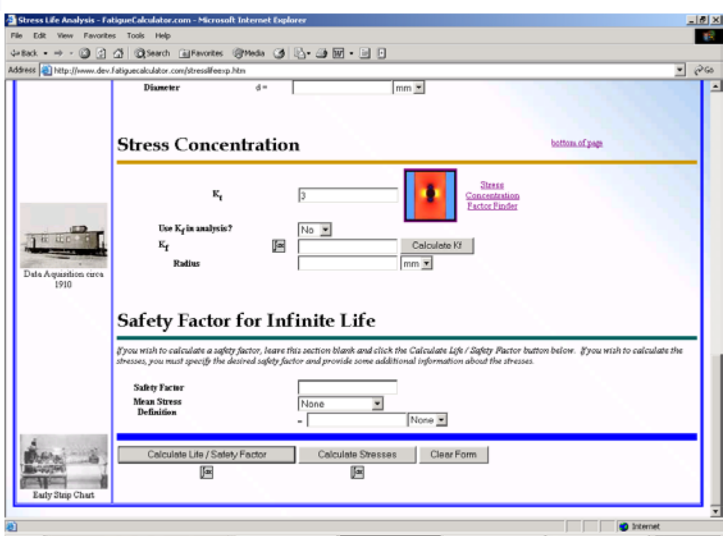
Deterministic Analysis



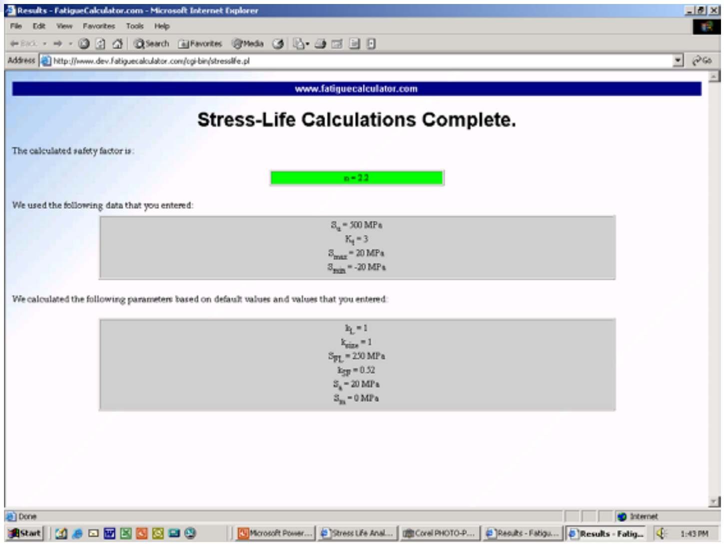
Deterministic Analysis (continued)



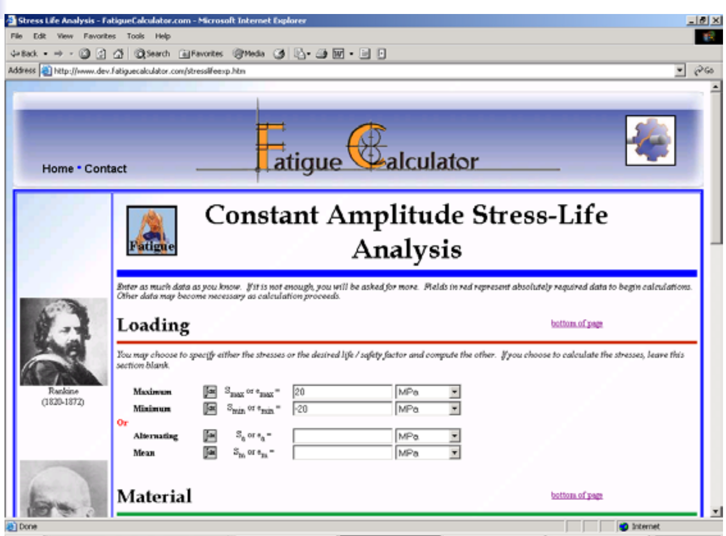
Deterministic Analysis (continued)



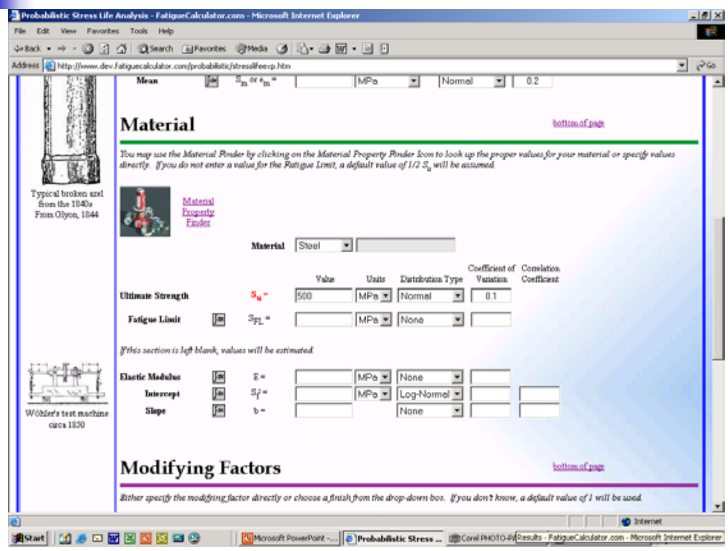
Deterministic Analysis Results



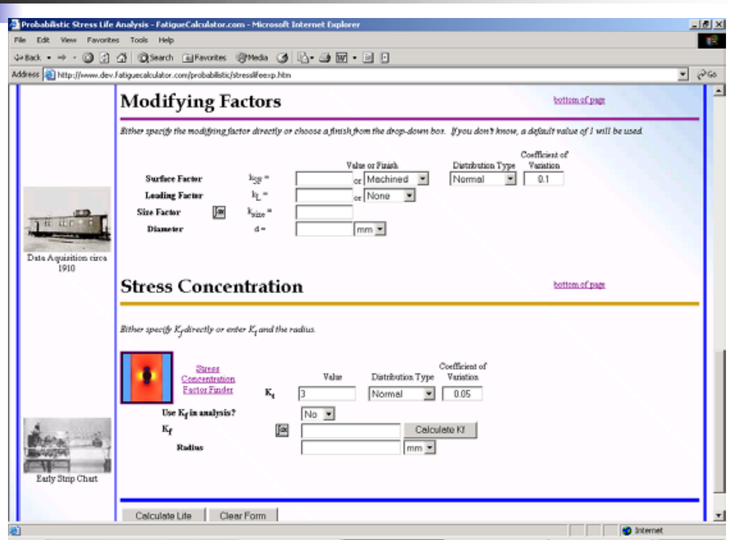
Probabilistic Analysis



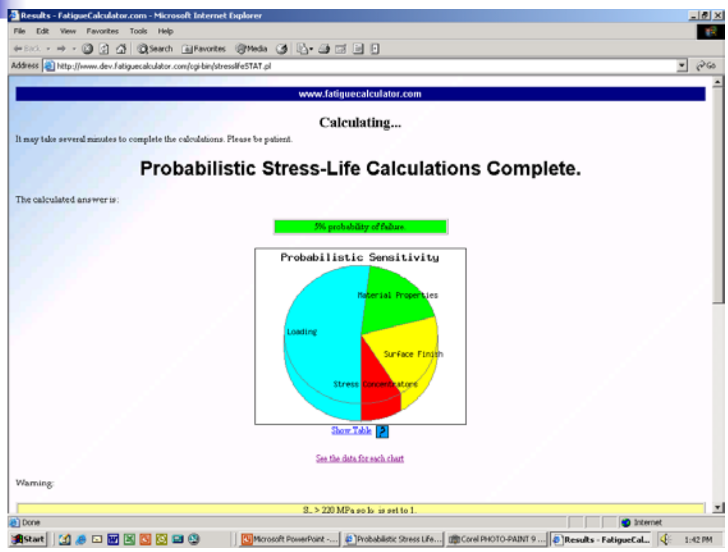
Probabilistic Analysis (continued)



Probabilistic Analysis (continued)



Probabilistic Analysis Results



Probabilistic Aspects of Fatigue

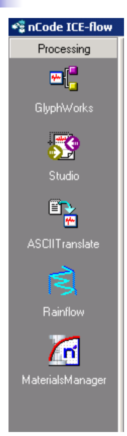
GlyphWorks

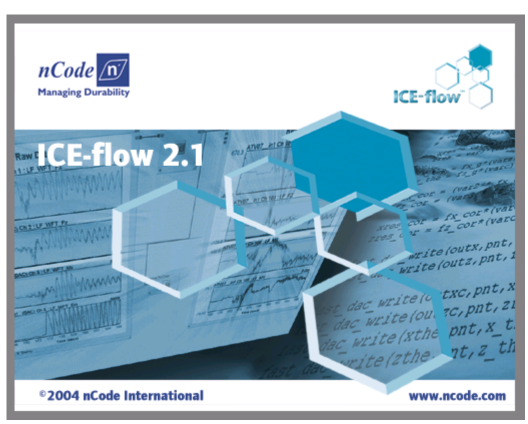
Professor Darrell F. Socie
Department of Mechanical and Industrial Engineering

Probabilistic Aspects of Fatigue

- Introduction
- Basic Probability and Statistics
- Statistical Techniques
- Analysis Methods
- Characterizing Variability
- Case Studies
- FatigueCalculator.com
- **GlyphWorks**

GlyphWorks / Rainflow





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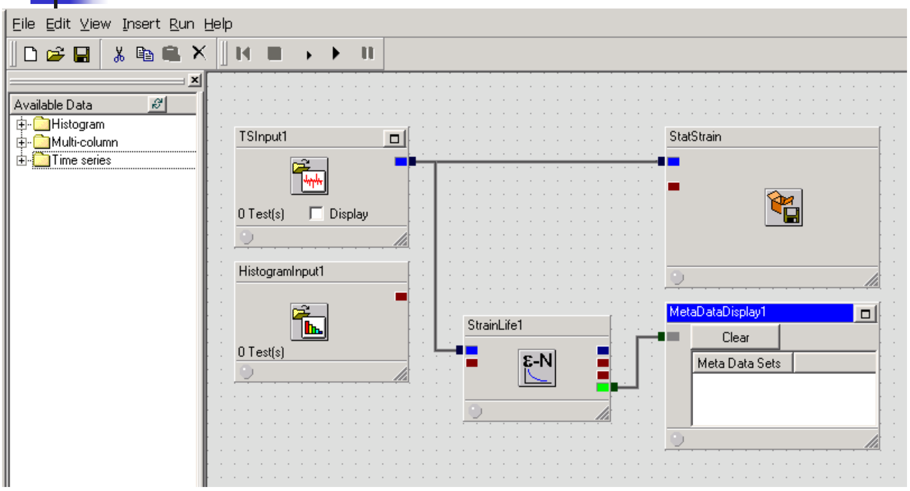
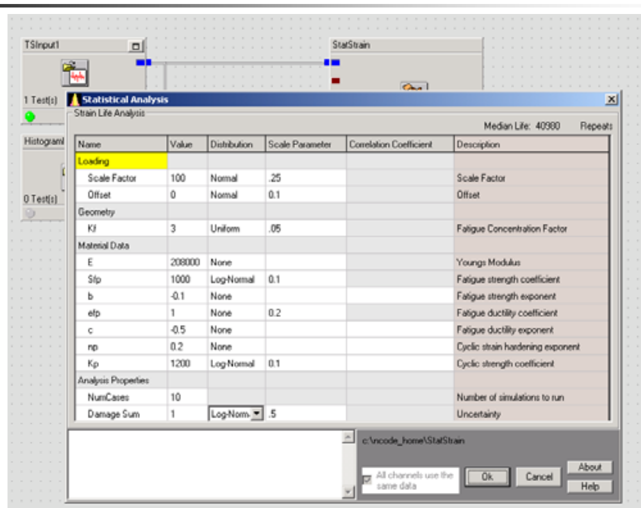
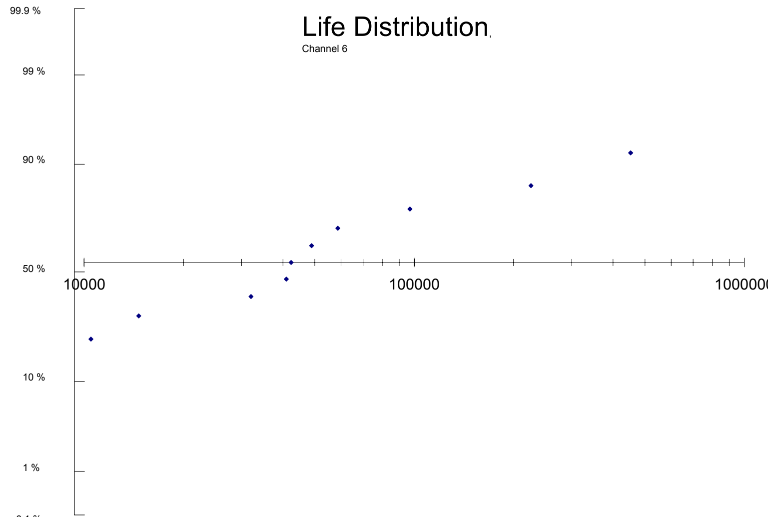
StatStrain.flo



StatStrain Glyph

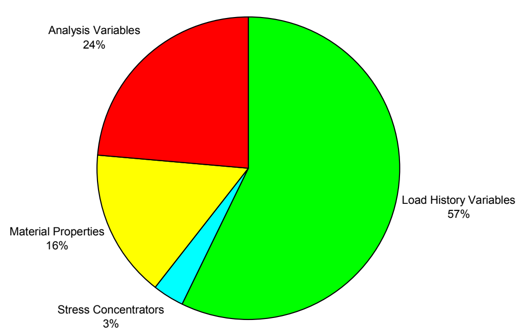


StatOutput.xls



StatOutput.xls (continued)

Probabilistic Sensitivity, Channel 6



StatOutput.xls (continued)

Channel # 6 Total Channels 1

Life Distribution

NumCases	11
Median	45634
COV	1.91
Probability (%)	Life
99	817179
90	223648
50	45634
10	9311
1	2548

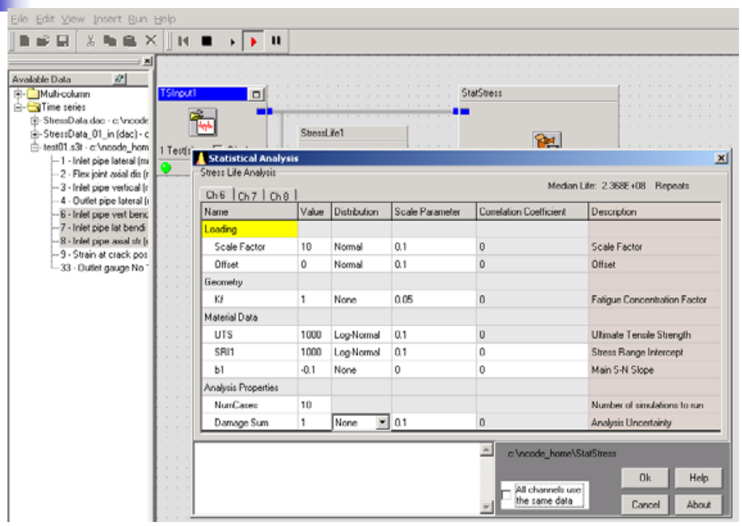
Sensitivity

Variable	Probabilistic	Deterministic
Load History Variables	0.89	-5.42
ScaleFactor	0.89	-5.42
Offset	0.00	0.00
Stress Concentrators	0.05	-5.51
Kf	0.05	-5.51
Material Properties	0.25	-14.54
E	0.00	-4.37
Sfp	0.23	3.42
b	0.00	-4.39
efp	0.00	1.42
c	0.00	-9.86
np	0.00	-2.32
Kp	0.10	1.56
Analysis Variables	0.37	1.00
Uncertainty	0.37	1.00

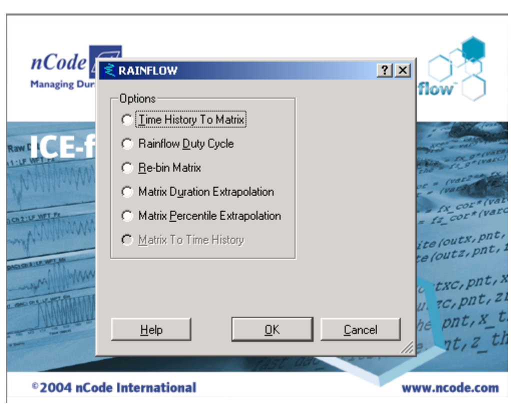
StatOutput.xls (continued)

		Inputs		Outputs	
Variable	Distribution	Value	Scale Parameter	Value	Scale Parameter
ScaleFactor	Normal	100	0.25	103	0.18
Offset	Normal	0	0.10	-0.01	0.08
Kf	Uniform	3	0.05	3.0	0.04
E	None	208000	0.00	208000	0.00
Sfp	Log-Normal	1000	0.10	1054	0.11
b	None	-0.1	0.00	-0.10	0.00
efp	None	1	0.20	1.0	0.00
c	None	-0.5	0.00	-0.50	0.00
np	None	0.2	0.00	0.20	0.00
Kp	Log-Normal	1200	0.10	1194	0.10
Uncertainty	Log-Normal	1	0.50	0.93	0.48

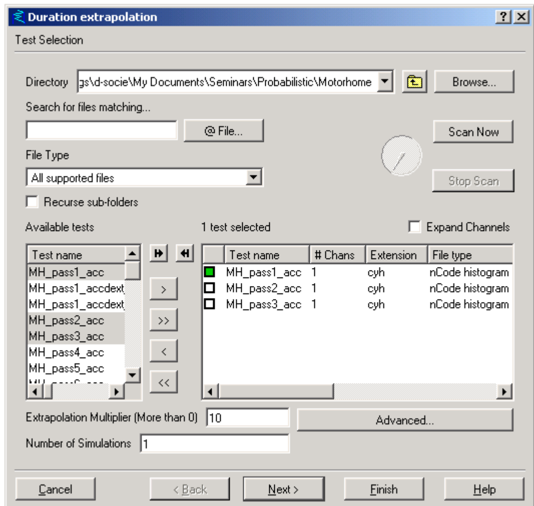
StatStress.flo



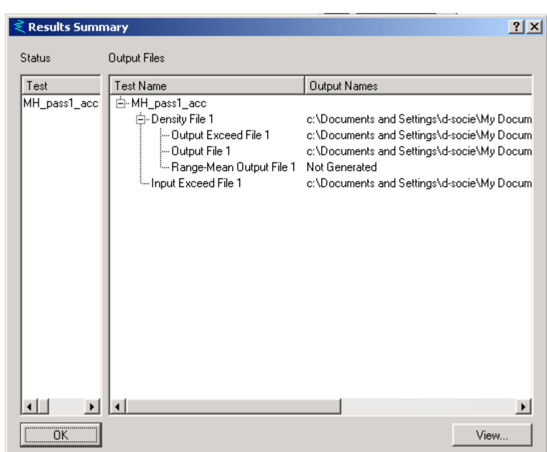
Rainflow Extrapolation



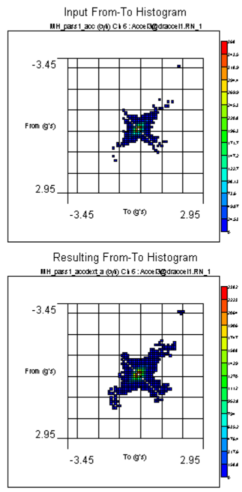
Duration Extrapolation



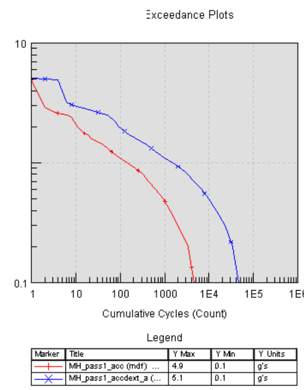
Results Files



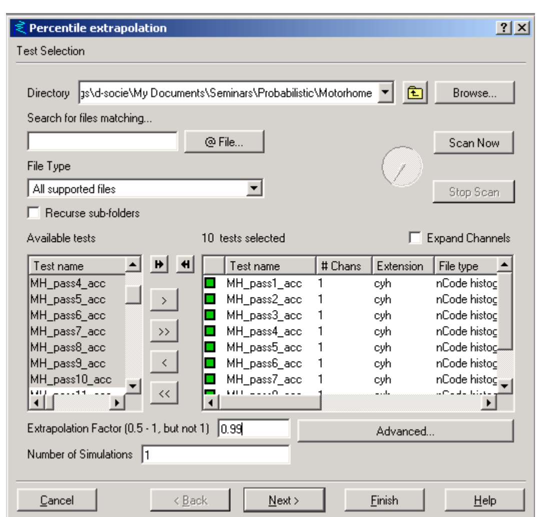
Rainflow Duration



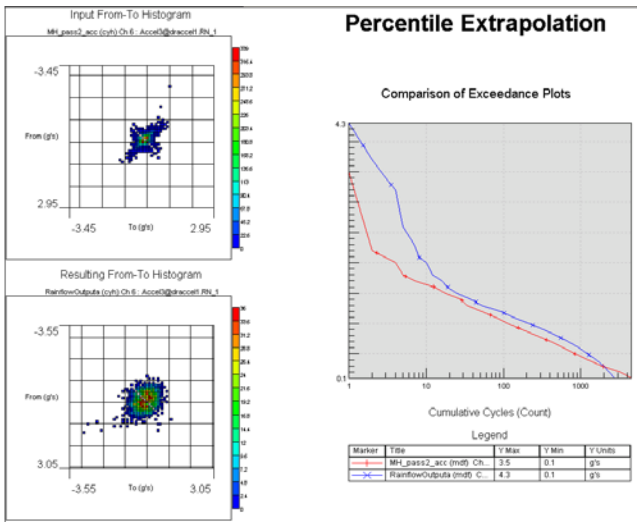
Duration Extrapolation



Percentile Extrapolation



Results



Probabilistic Aspects of Fatigue