Fatigue and Fracture

Static Strength and Fracture
Stress Concentration Factors

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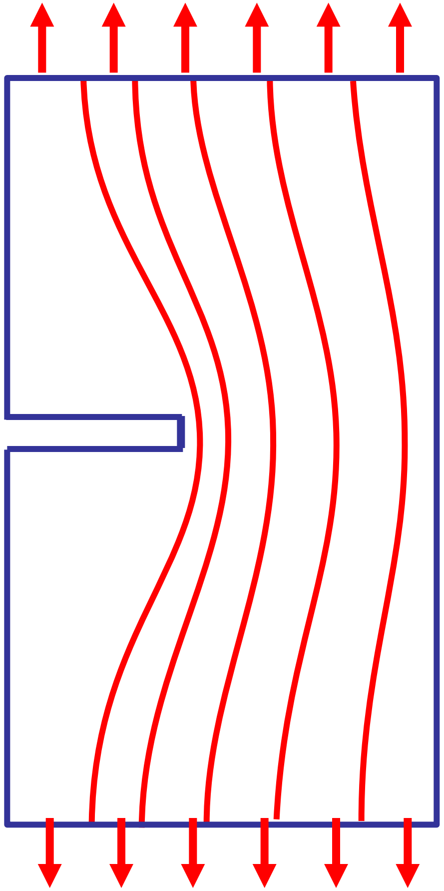
Static Strength and Fracture

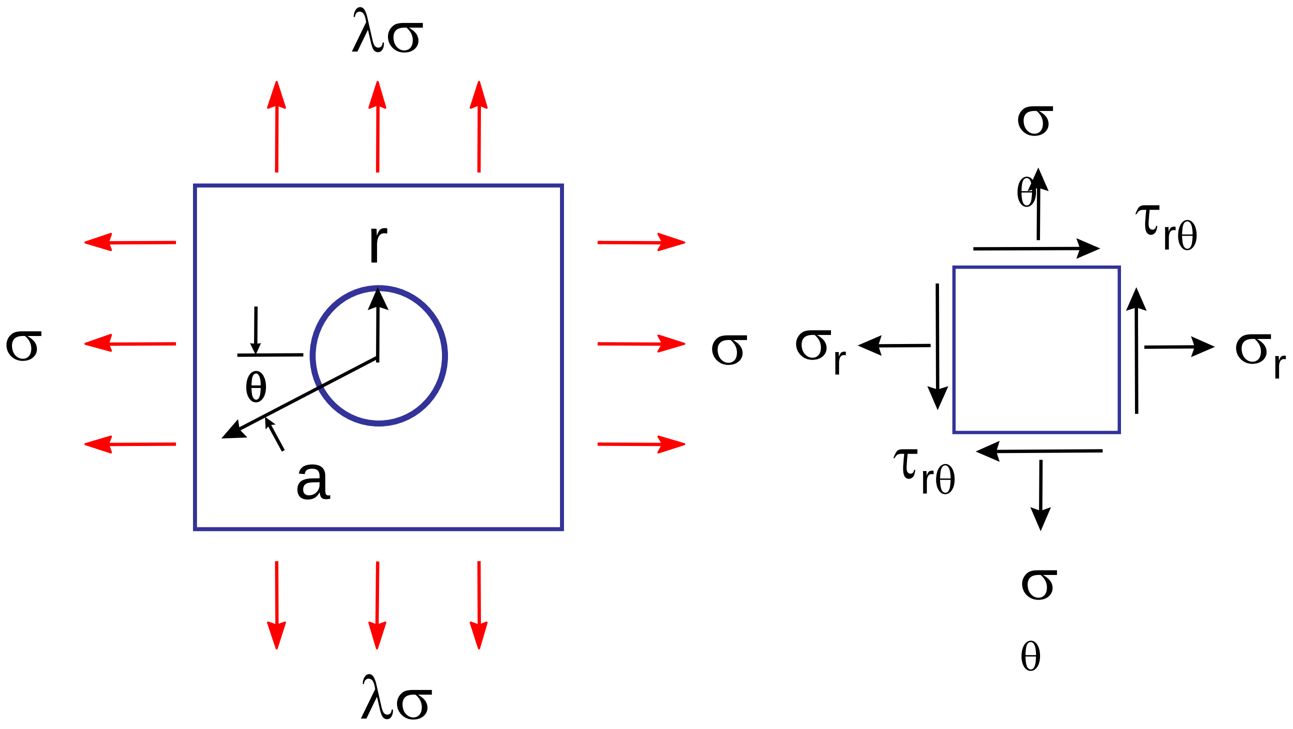
- **Stress Concentration Factors**
- Fracture Mechanics
- Approximate Stress Intensity Factors
- Ductile vs. Brittle Fracture

Stress Concentration



“Load flow” lines

Circular Notch



Stresses

$$\frac{\sigma_r}{\sigma}=\frac{1+\lambda}{2}\left(1-\left(\frac{r}{a}\right)^2\right)+\frac{1-\lambda}{2}\left(1+3\left(\frac{r}{a}\right)^4-4\left(\frac{r}{a}\right)^2\right)\cos 2\theta$$

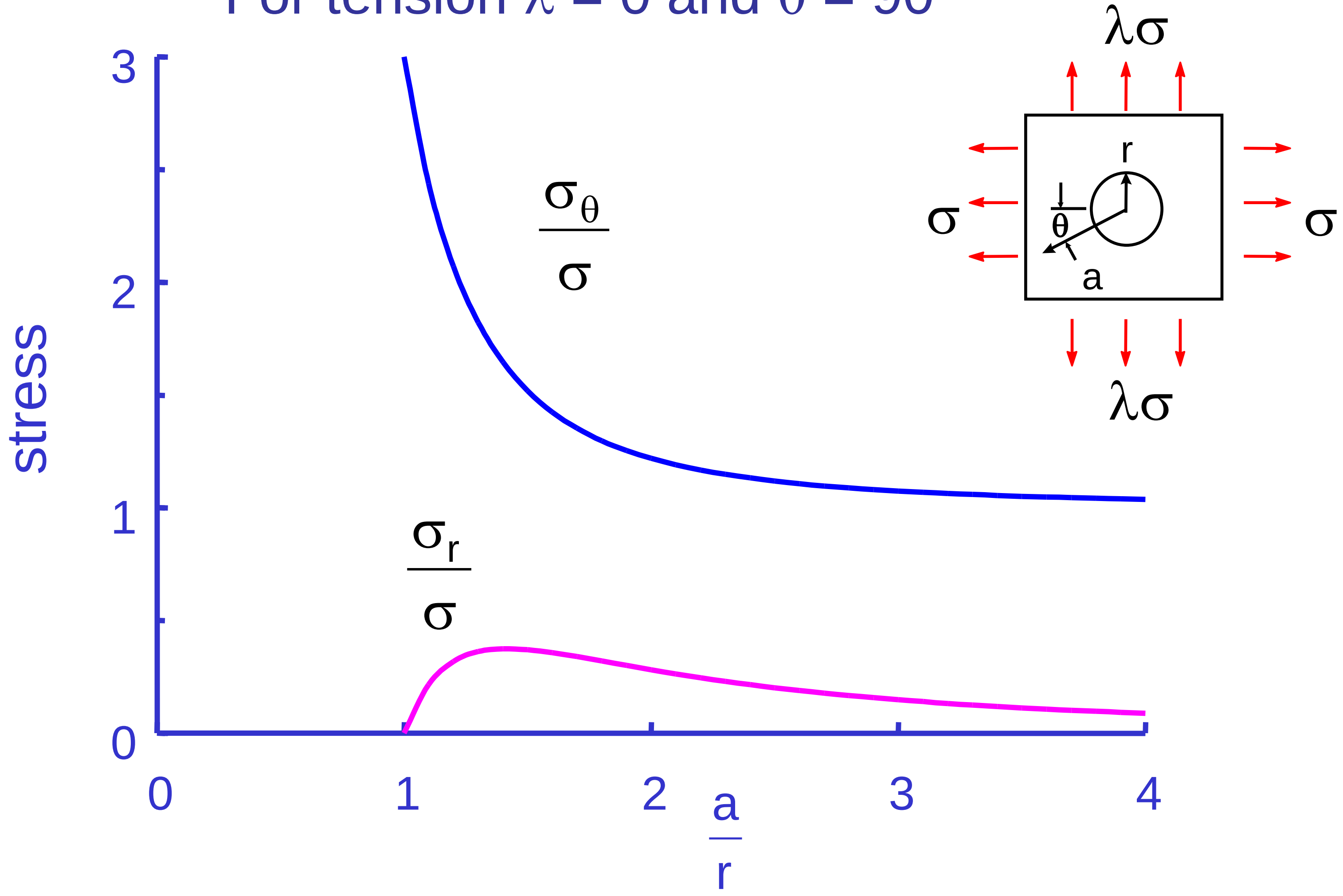
$$\frac{\sigma_\theta}{\sigma}=\frac{1+\lambda}{2}\left(1+\left(\frac{r}{a}\right)^2\right)-\frac{1-\lambda}{2}\left(1+3\left(\frac{r}{a}\right)^4\right)\cos 2\theta$$

$$\frac{\tau_{r\theta}}{\sigma}=-\frac{1-\lambda}{2}\left(1-3\left(\frac{r}{a}\right)^4+2\left(\frac{r}{a}\right)^2\right)\sin 2\theta$$

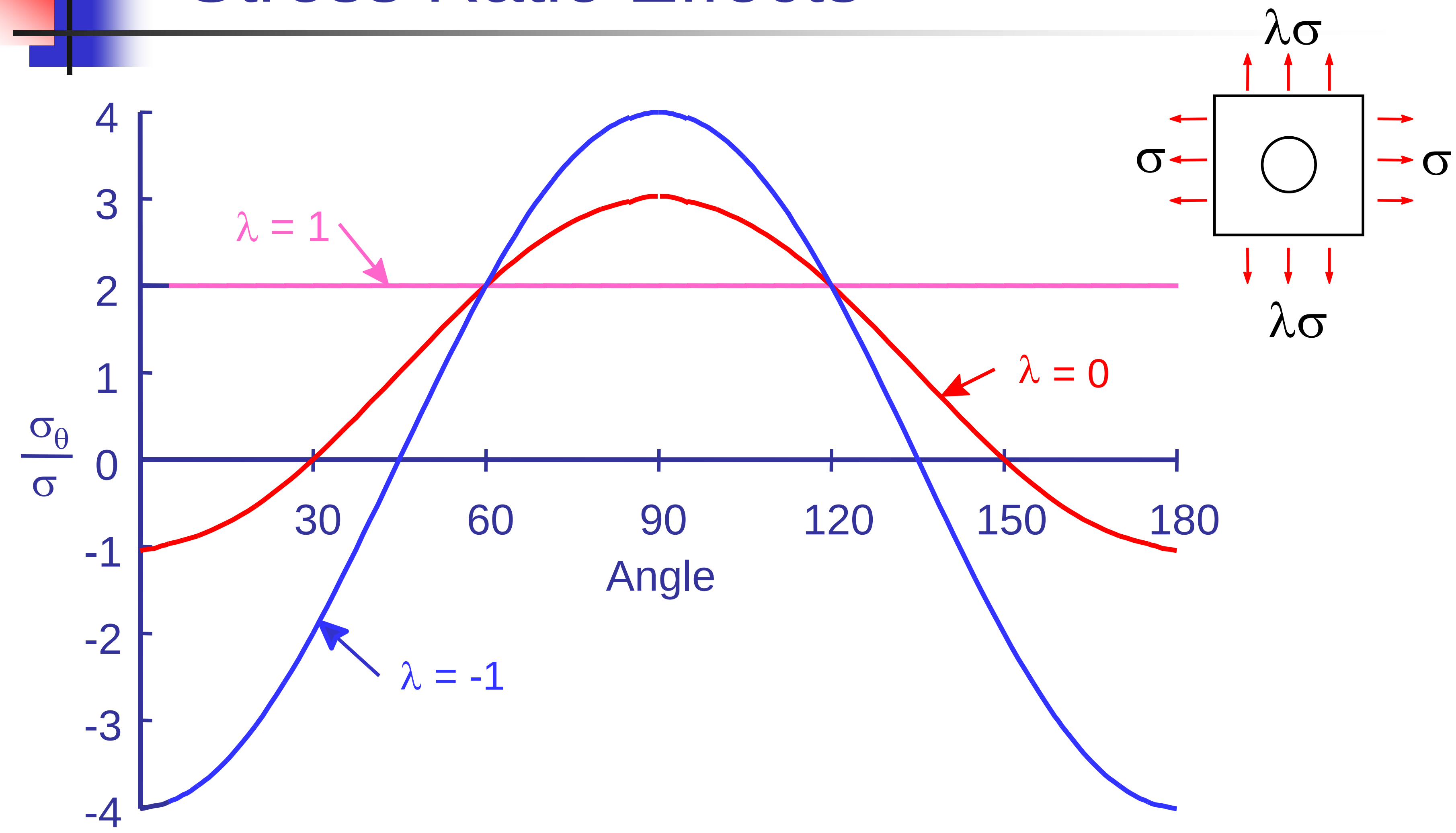
Independent of size, dependant only on r/a

Stress Distribution

For tension λ = 0 and θ = 90

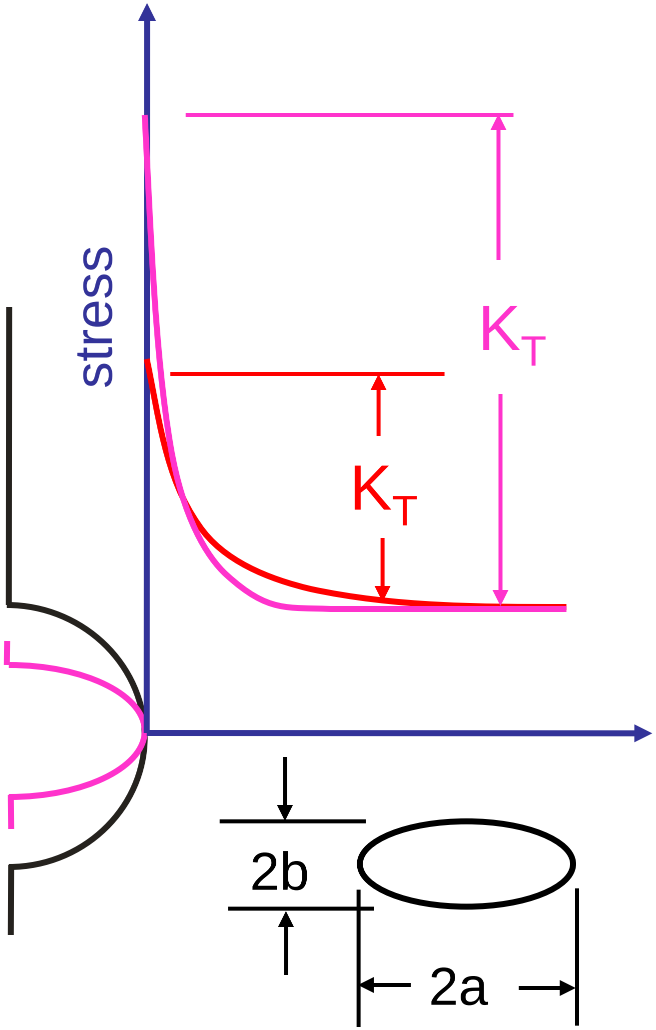


Stress Ratio Effects



stresses around the circumference of a hole

Elliptical Notches

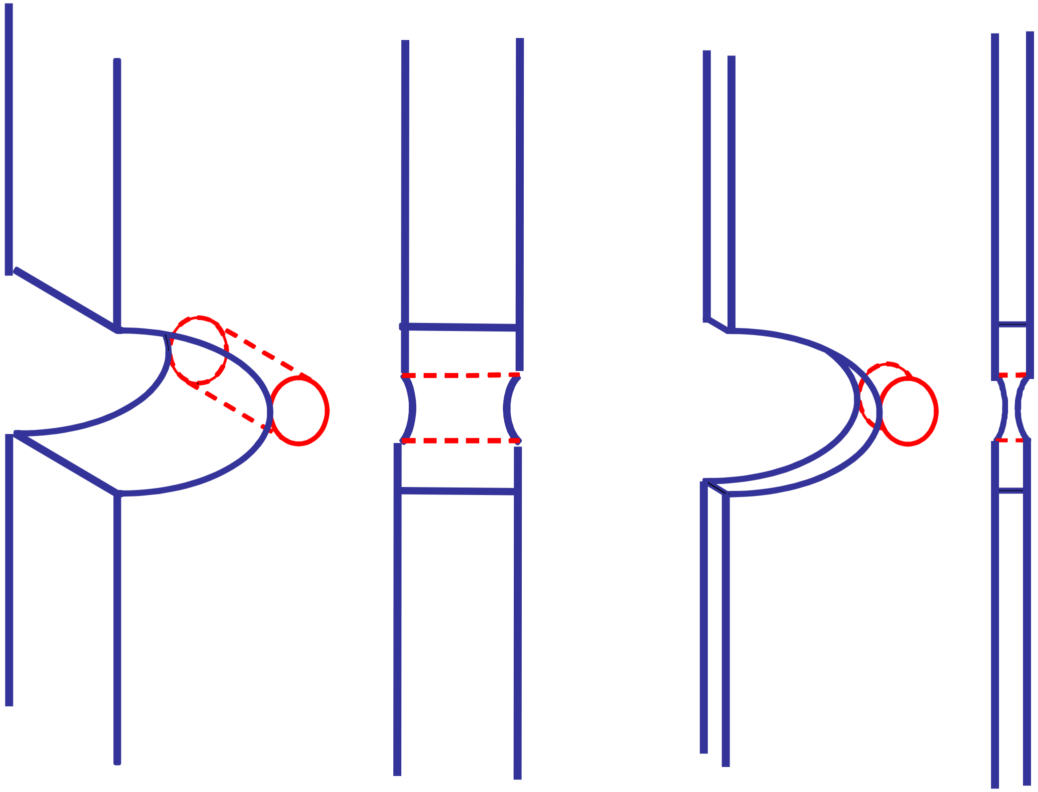


$$K_T = 1 + 2\sqrt{\frac{a}{\rho}} \qquad \rho = \frac{b^2}{a}$$

Sharp Notch:
- high K_T
- high gradient

Blunt Notch:
- low K_T
- low gradient

Plane Stress – Plane Strain



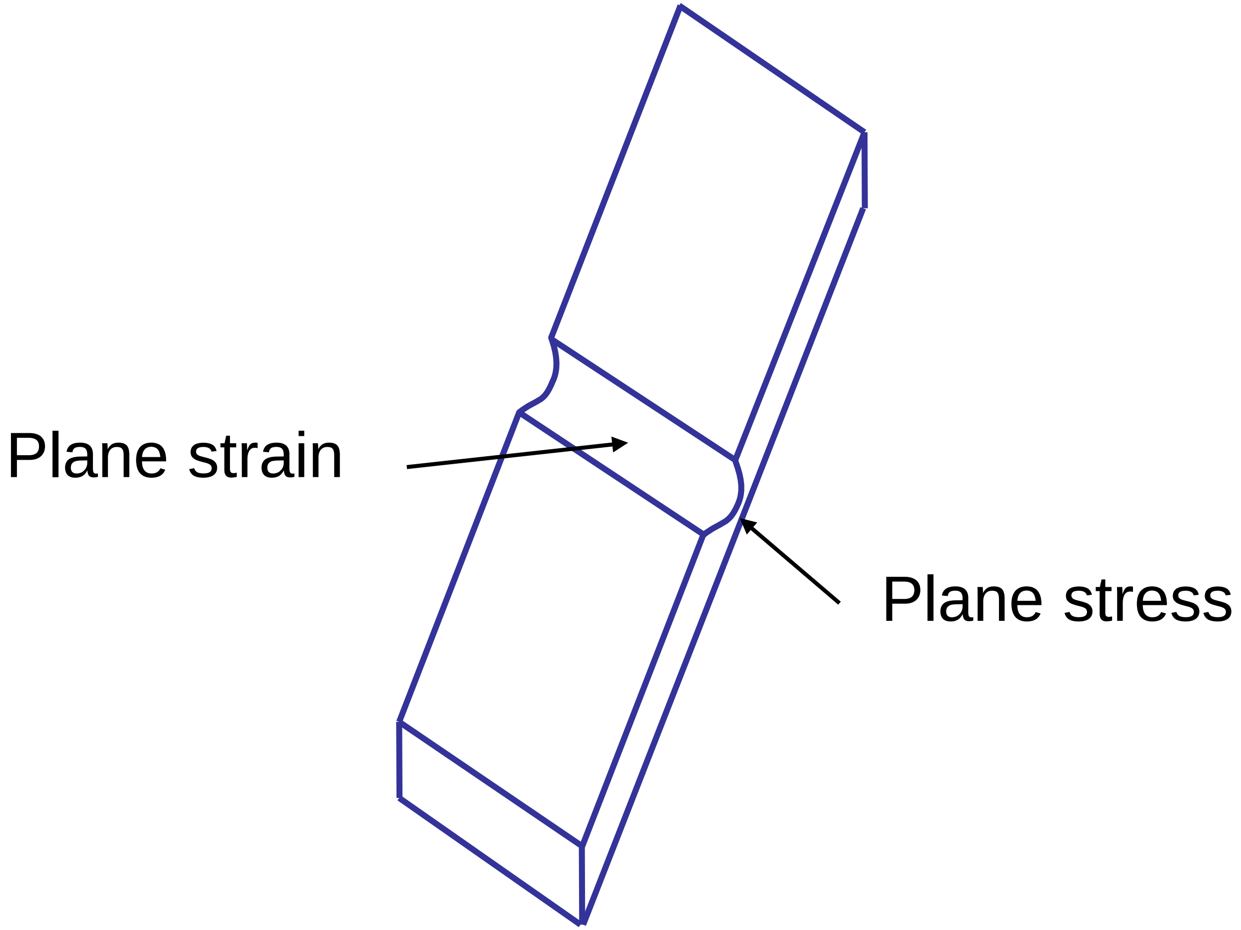
thick plate

thin plate

plane strain $\sigma_z \neq 0$ $\varepsilon_z = 0$

plane stress $\sigma_z = 0$ $\varepsilon_z \neq 0$

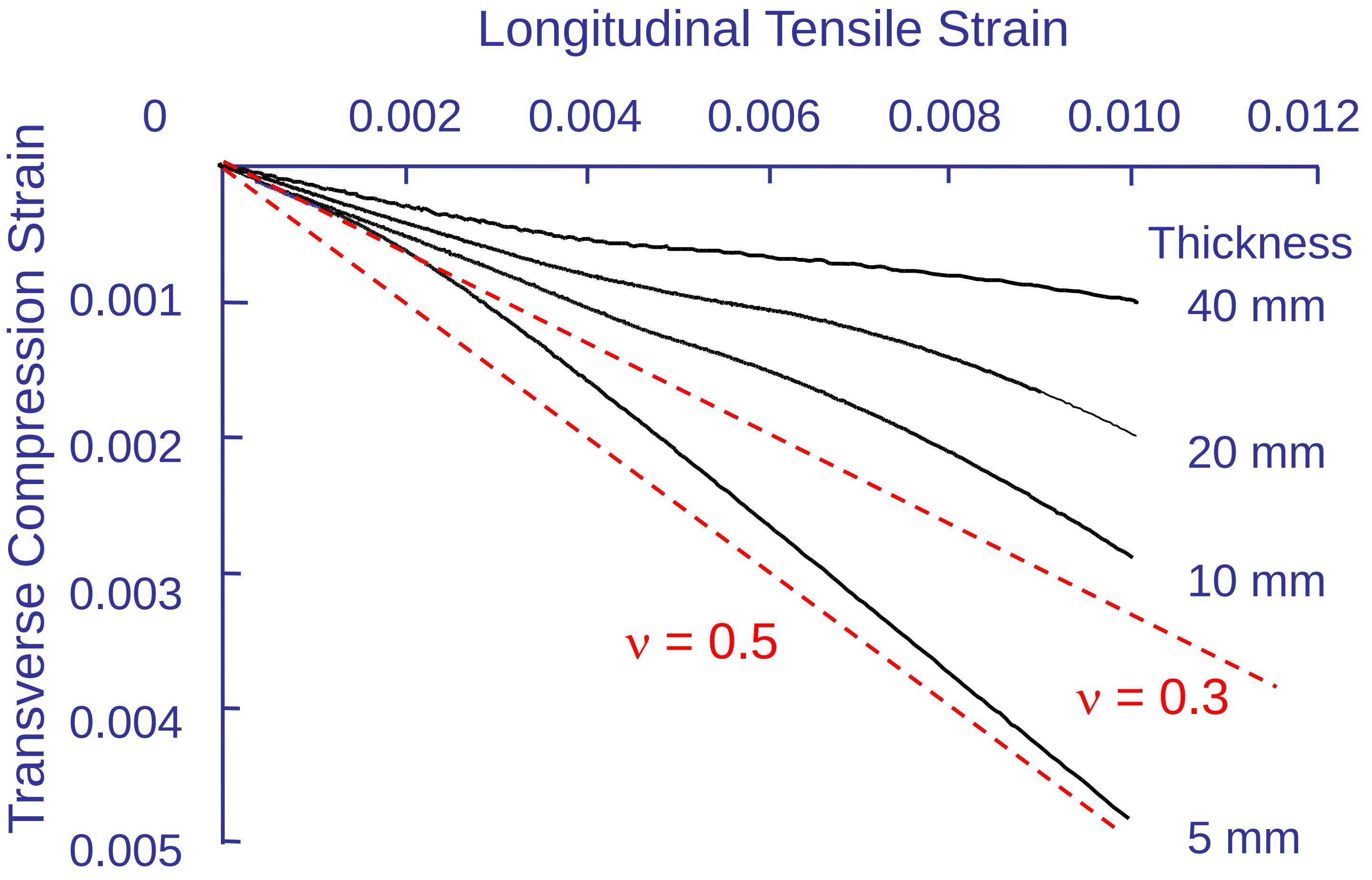
Thick or Thin ?

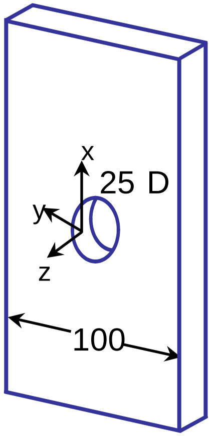


Plane strain

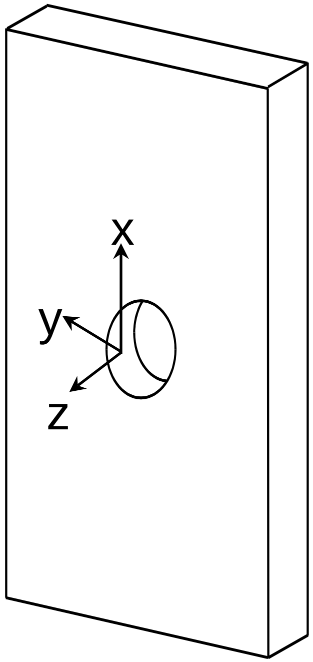
Plane stress

Transverse Strains





Notch Stresses

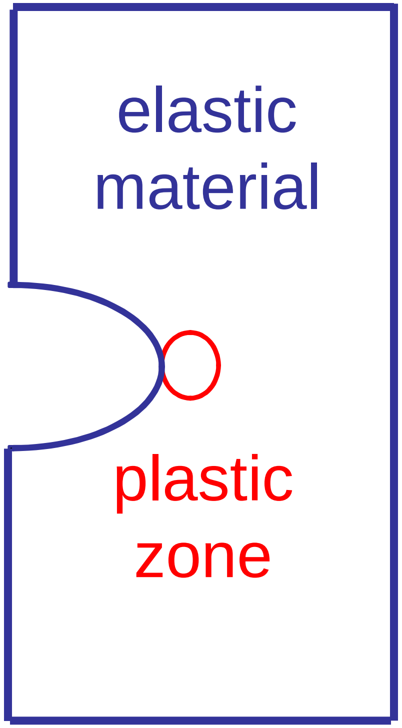


t	ε_x	ε_z	σ_x	σ_z
7	0.01	-0.005	63.5	0
15	0.01	-0.003	70.6	14.1
30	0.01	-0.002	73.0	21.8
50	0.01	-0.001	75.1	29.3

Fracture Surfaces

Fracture Surfaces

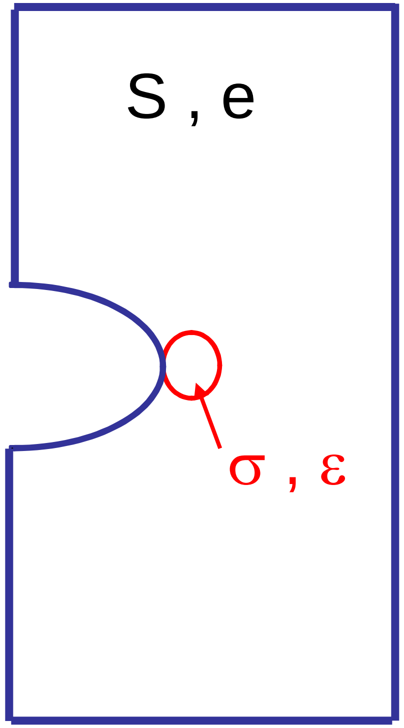
Stress or Strain Control?



Elastic material surrounding the plastic zone forces the displacements to be compatible, I.e. no gaps form in the structure.

Boundary conditions acting on the plastic zone boundary are displacements. Strains are the first derivative of displacement

Define K_σ and K_ε



Define: nominal stress, S

nominal strain, e

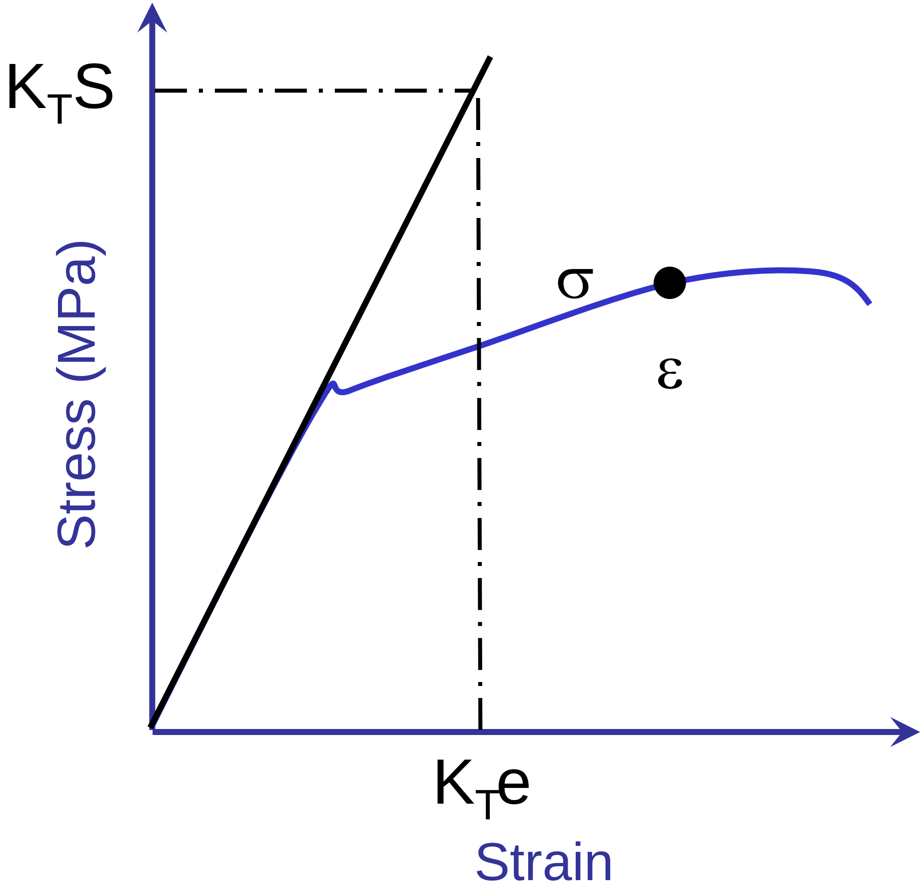
notch stress, σ

notch strain, ε

Stress concentration $K_\sigma = \frac{\sigma}{S}$

Strain concentration $K_\varepsilon = \frac{\varepsilon}{e}$

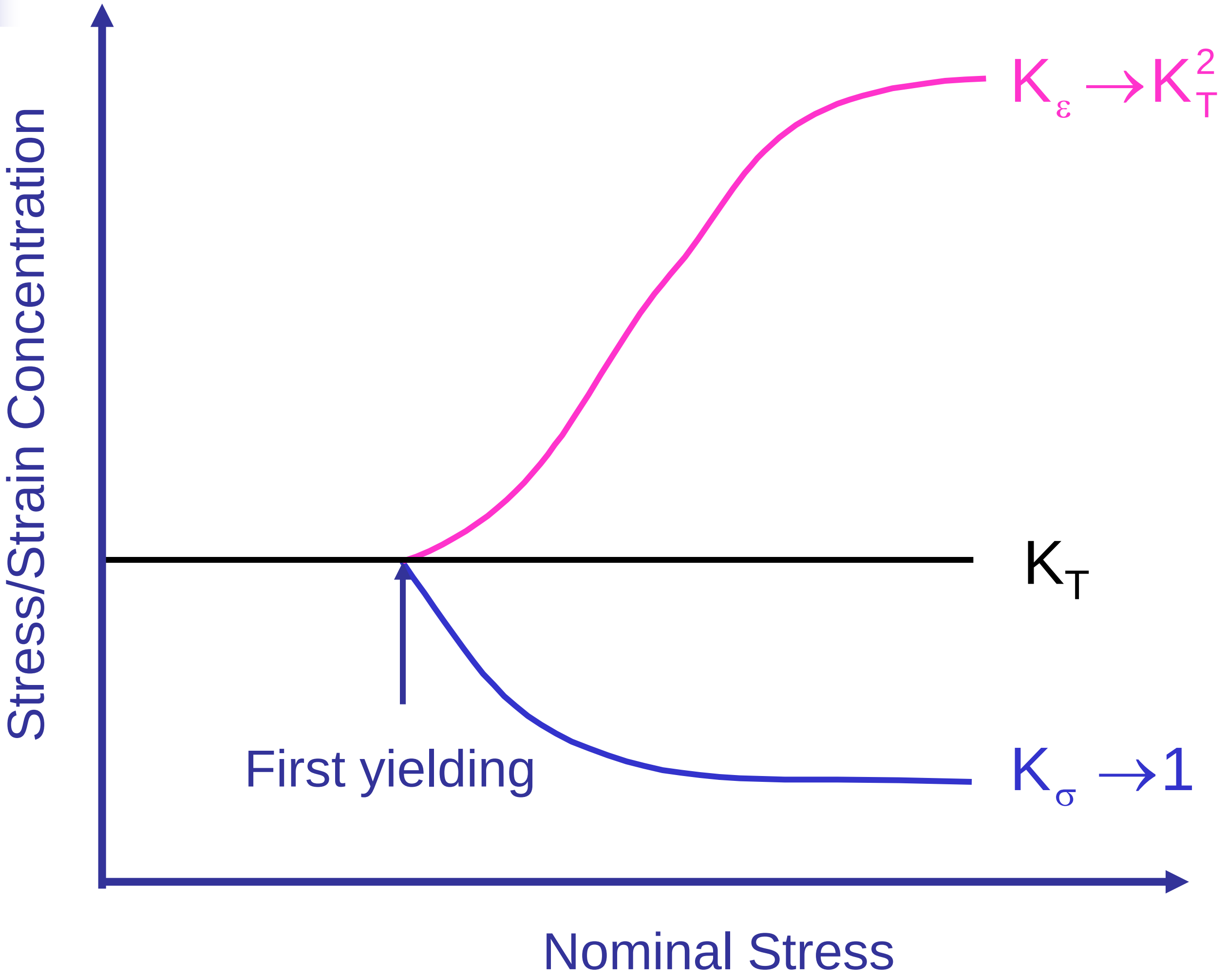
K_σ and K_ε



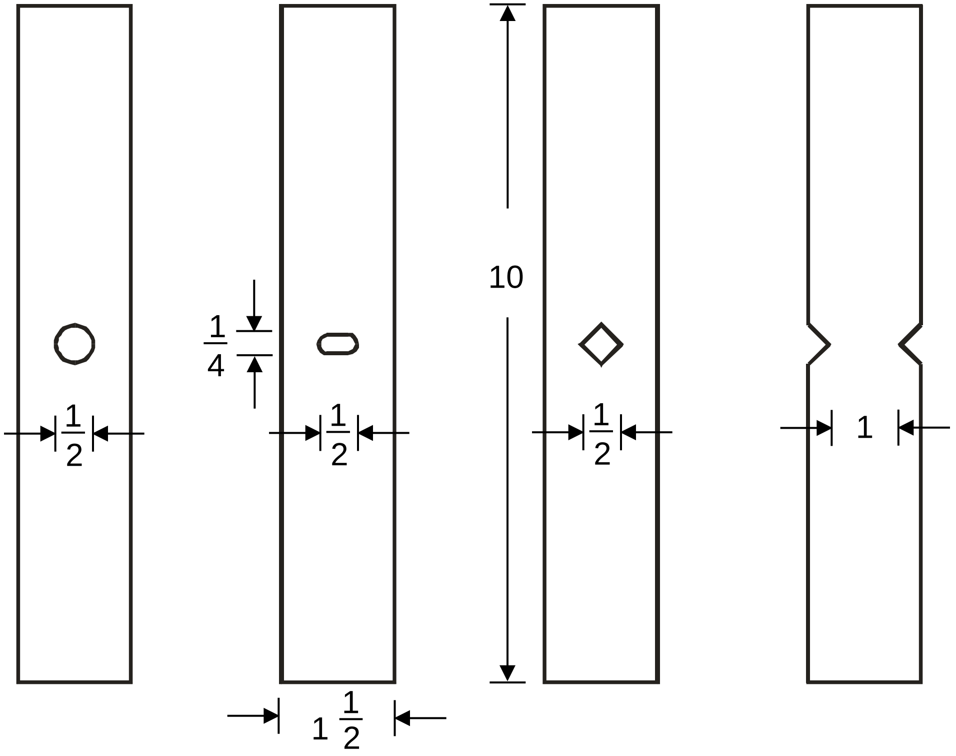
$$K_\sigma = \frac{\sigma}{S}$$

$$K_\varepsilon = \frac{\varepsilon}{e}$$

Stress and Strain Concentration



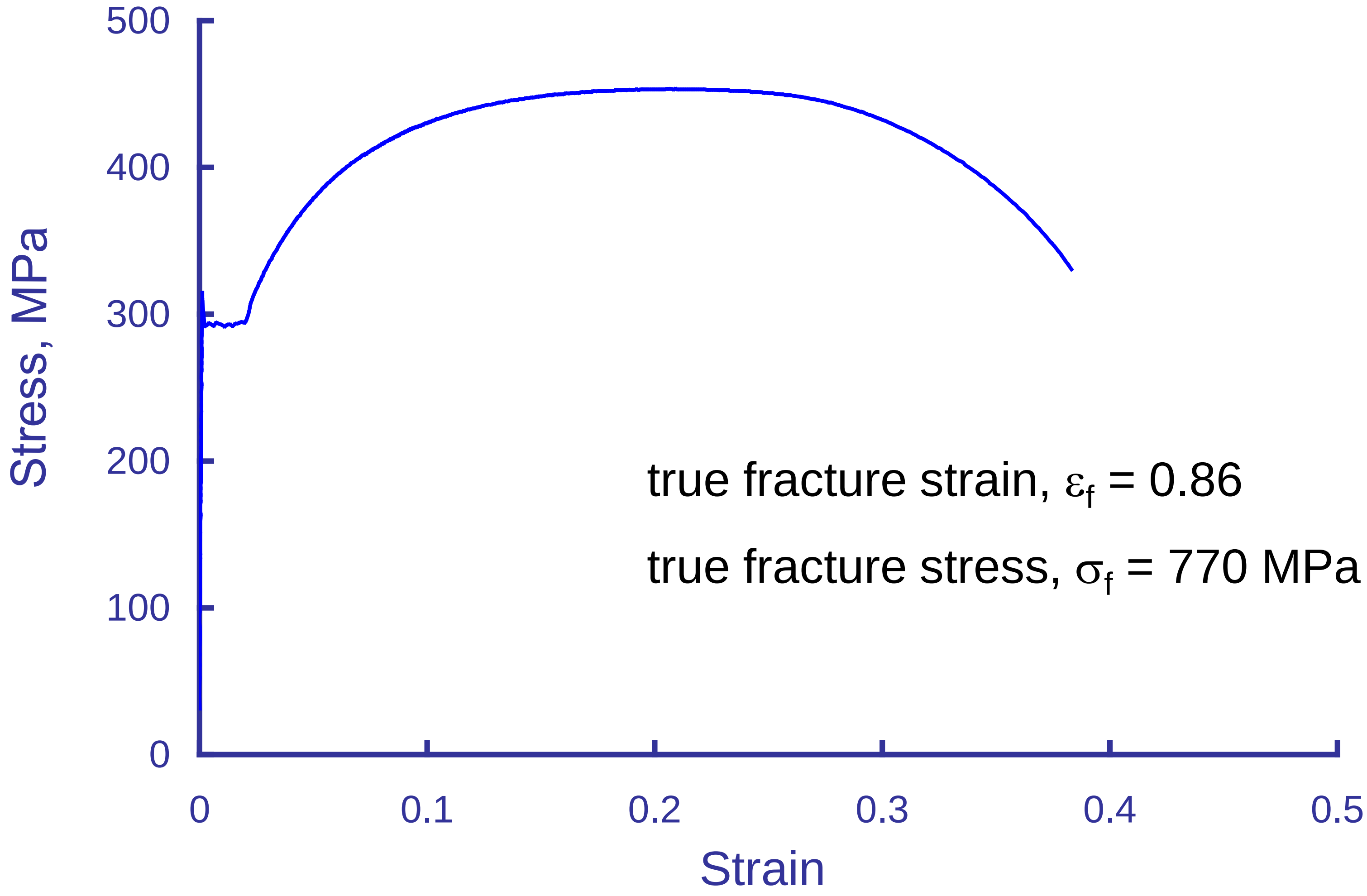
Notched Plate Experiments



Materials:
1018 Hot Rolled Steel
7075-T6 Aluminum

1/4 thick

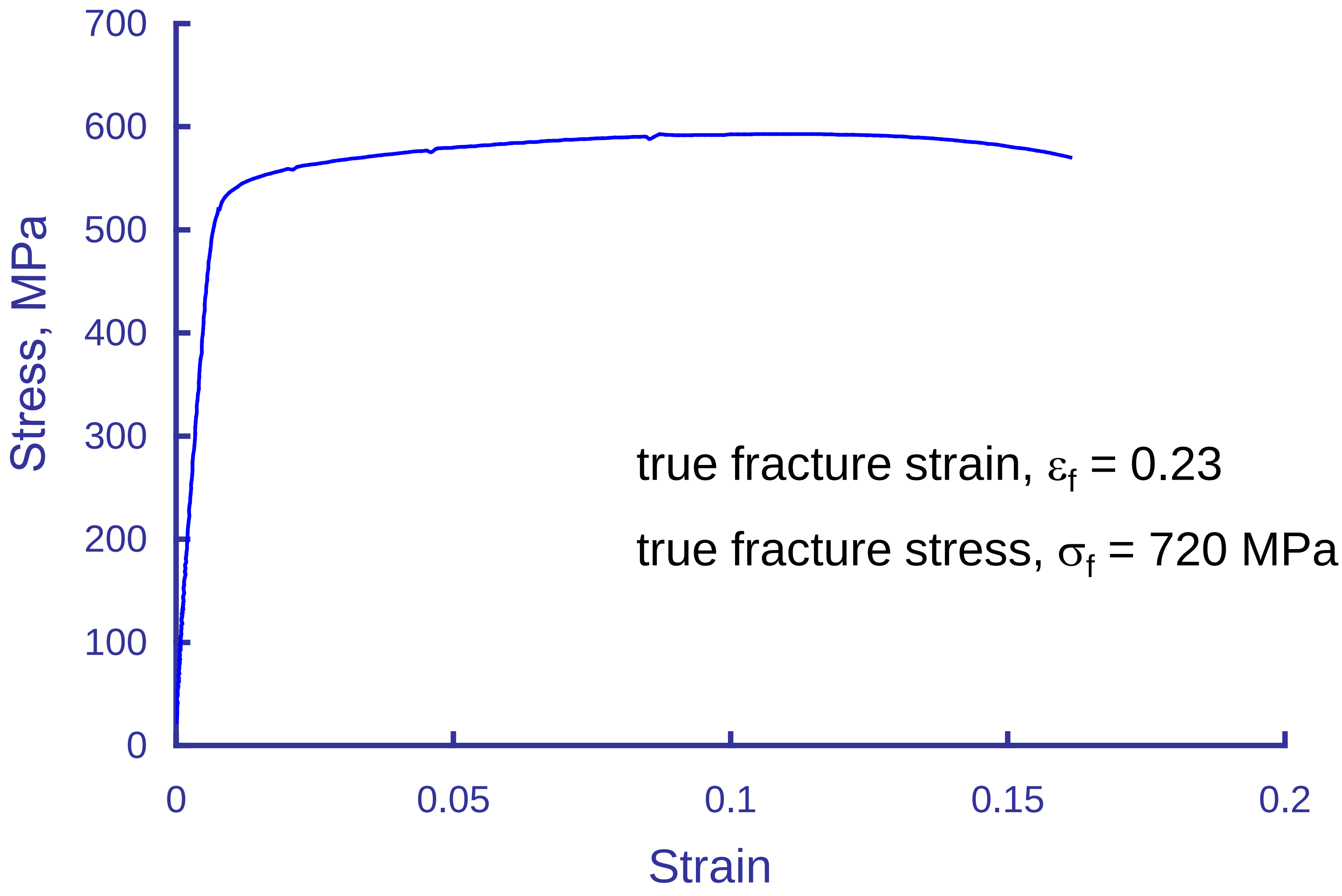
1018 Stress-Strain Curve



true fracture strain, $\varepsilon_f = 0.86$

true fracture stress, $\sigma_f = 770$ MPa

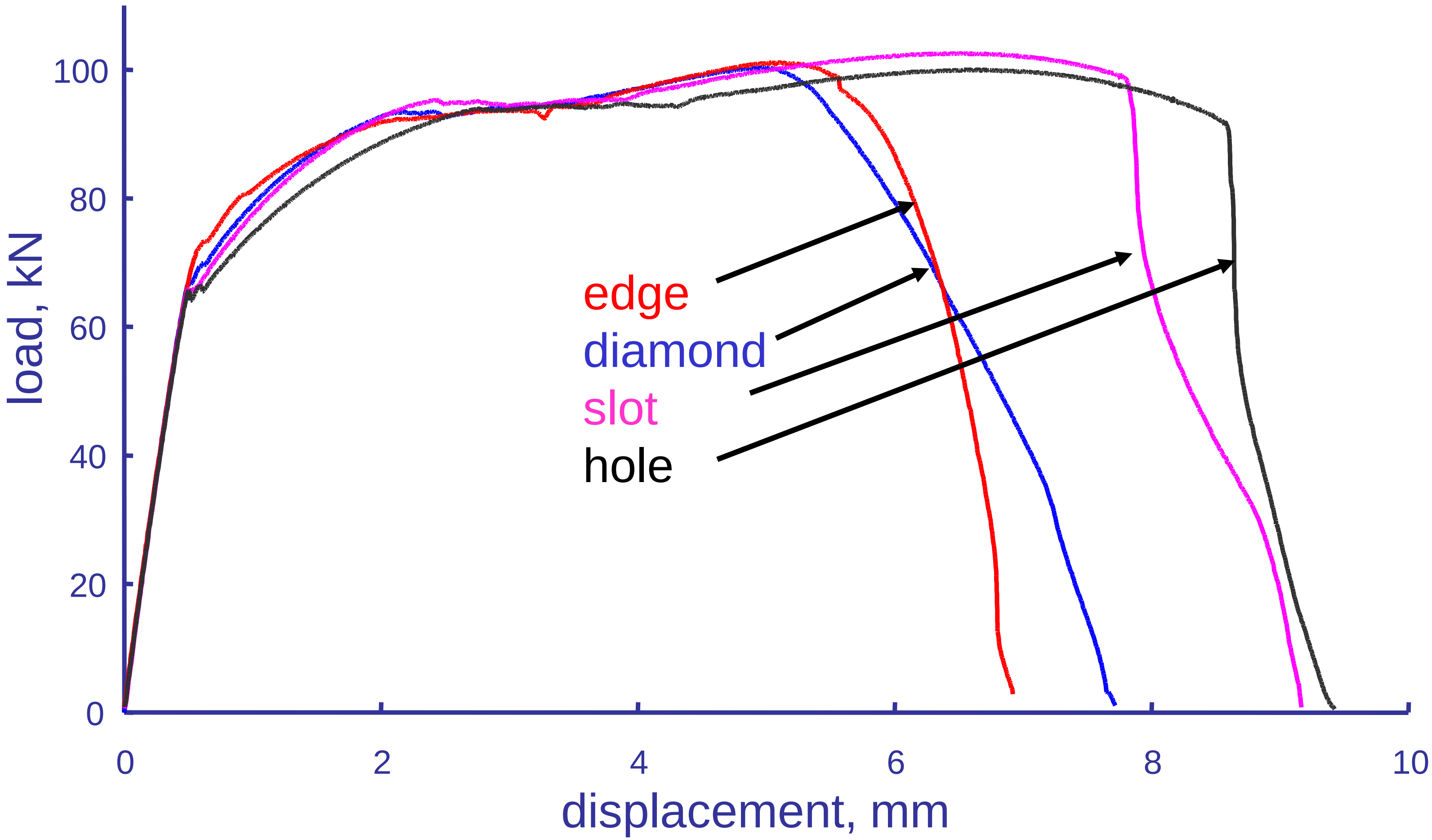
7075-T6 Stress-Strain Curve



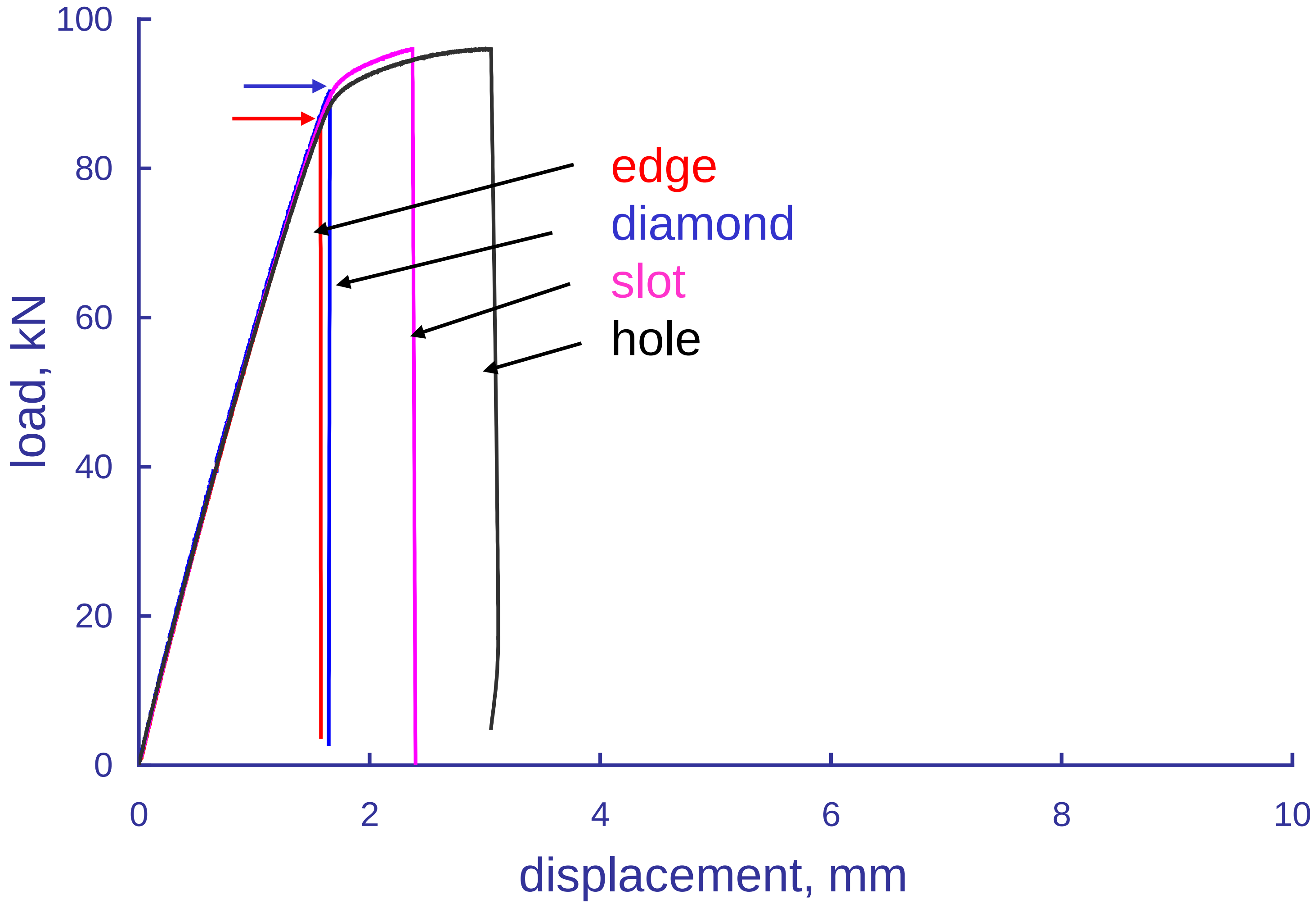
true fracture strain, $\varepsilon_f = 0.23$

true fracture stress, $\sigma_f = 720$ MPa

1018 Steel Test Data



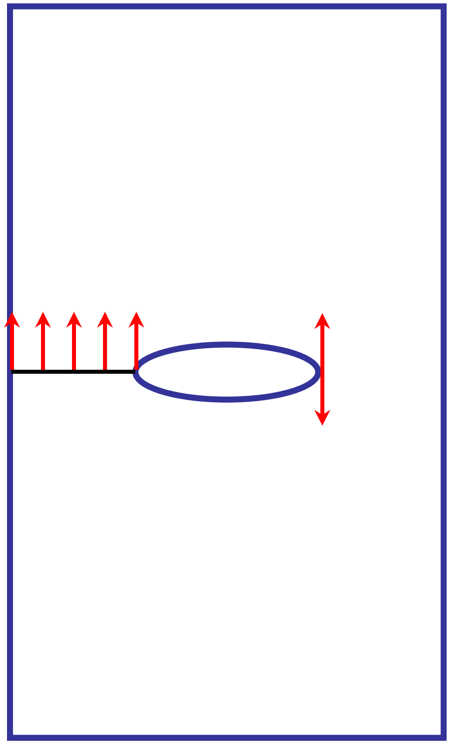
7075-T6 Test Data



Failure of a Notched Plate

Failures from Stress Concentrations

Net section stresses must be below the flow stress

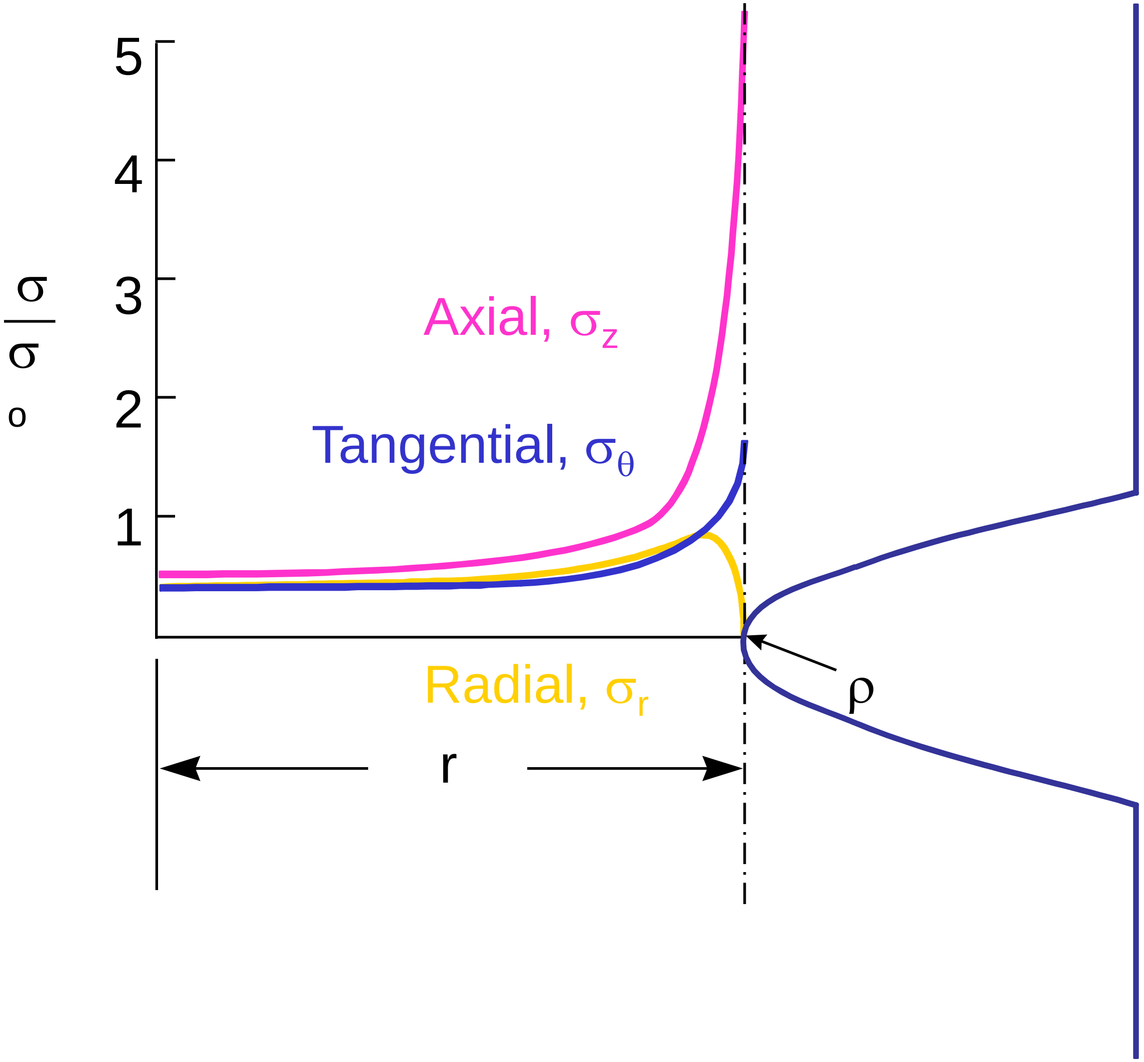


Notch strains must be below the fracture strain

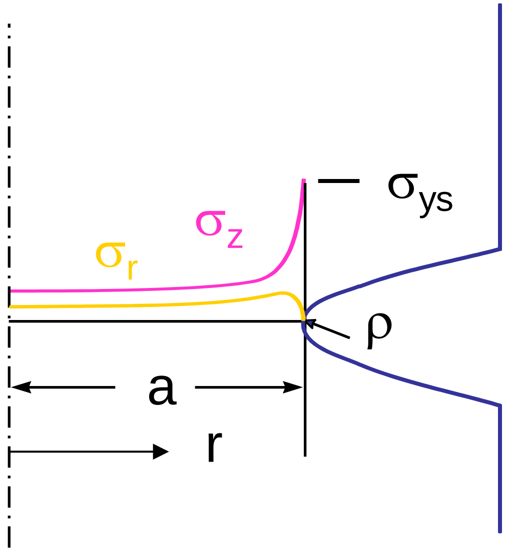
2D vs 3D

- Plates and shells
  - 2D stress state
- Solids
  - 3D stress state

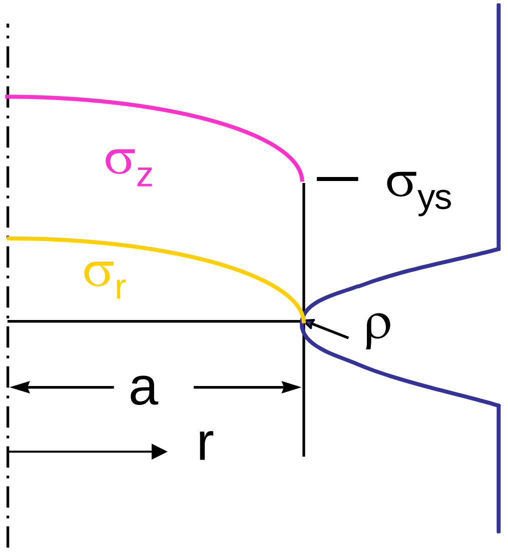
Stress Concentration in a Bar



Bridgeman Analysis (1943)



Elastic stress distribution



Plastic stress distribution

$$\tau = \frac{\sigma_z - \sigma_r}{2} = \mathrm{cons\,tan\,t}$$

Stresses

$$\sigma_z = \sigma_o \left[1 + \ln\left(\frac{a^2 + 2a\rho - r^2}{2a\rho} \right) \right]$$

$$P_z = \int_0^a 2\,\pi\, r\, \sigma_z\, dr$$

$$P_{max} = \pi\, a^2\, \sigma_{flow} \left(1 + \frac{2\rho}{a} \right) \ln\left(1 + \frac{a}{2\rho} \right)$$

$$P_{max} = A_{net}\, \sigma_{flow}\, CF$$

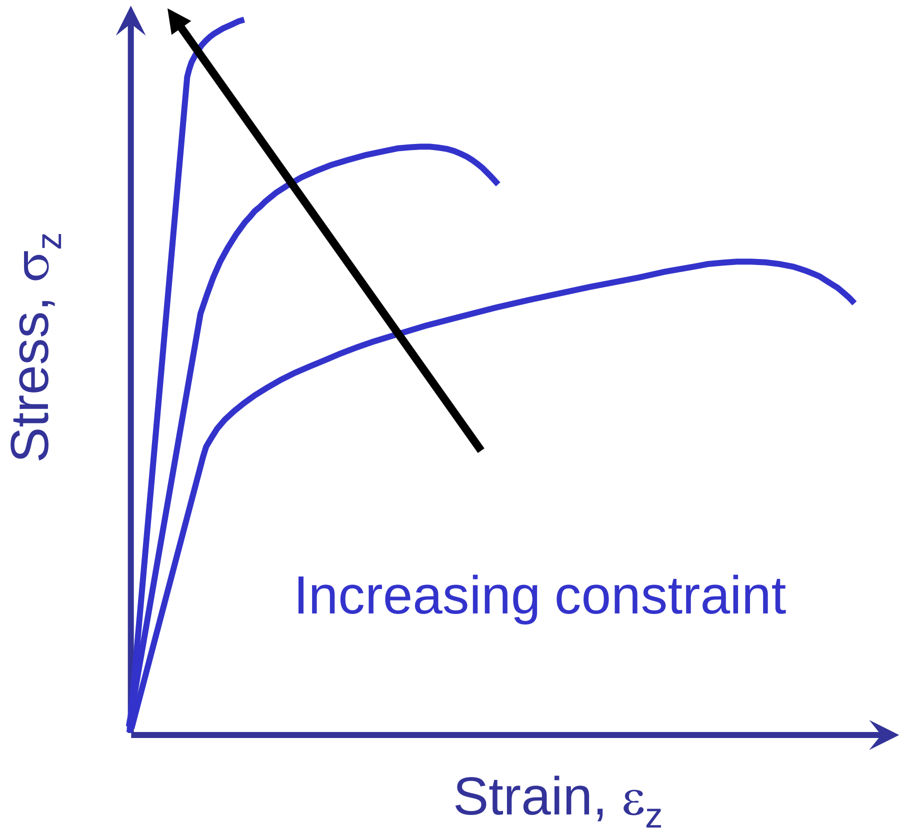
CF constraint factor

Constraint Factors

$a\,/\rho$	CF
0	1
1	1.21
2	1.38
4	1.64
8	1.73
20	2.63
∞	2.96

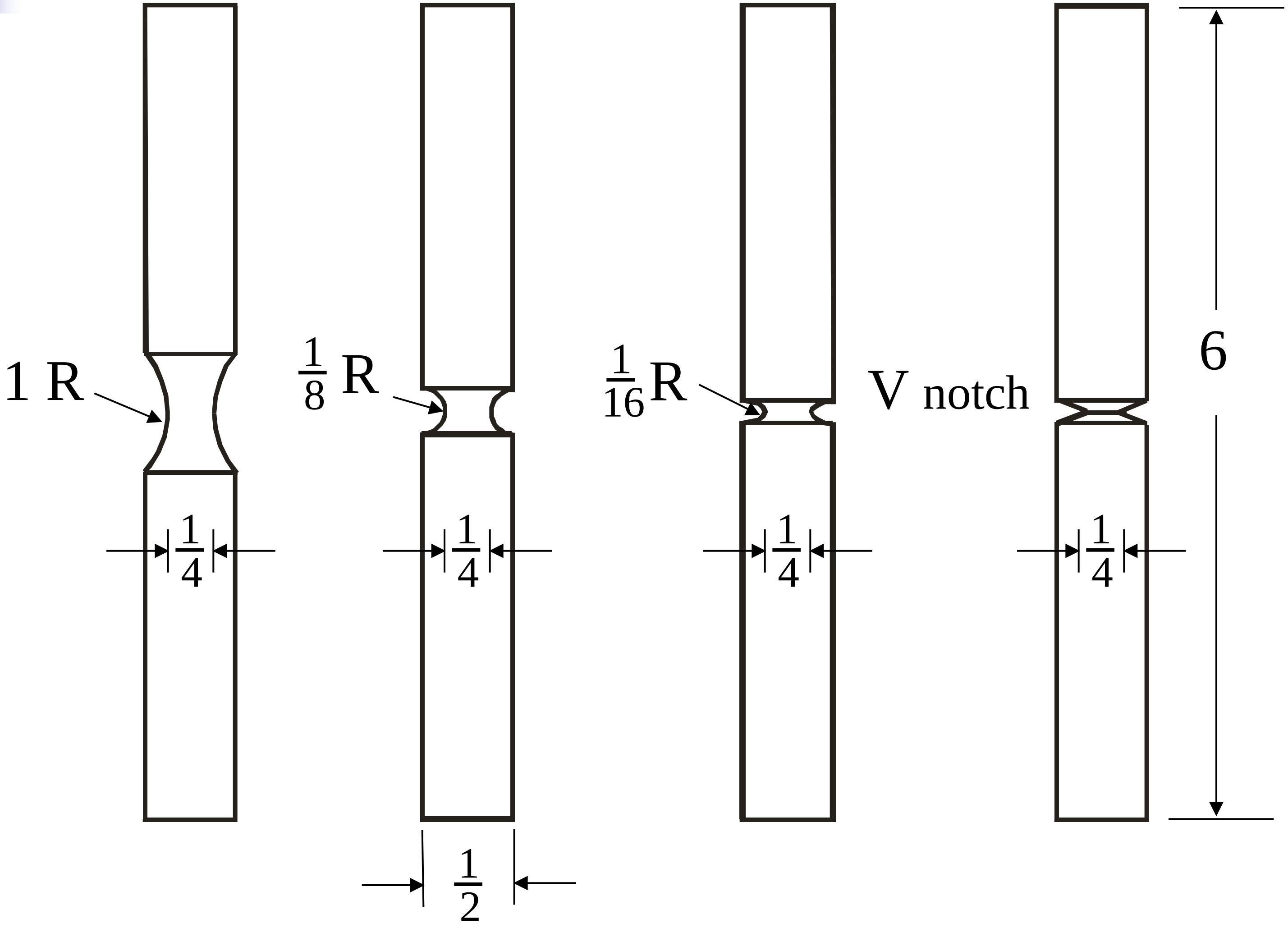
$$P_{max} = A_{net}\,\sigma_{flow}\,CF$$

Effect of Constraint

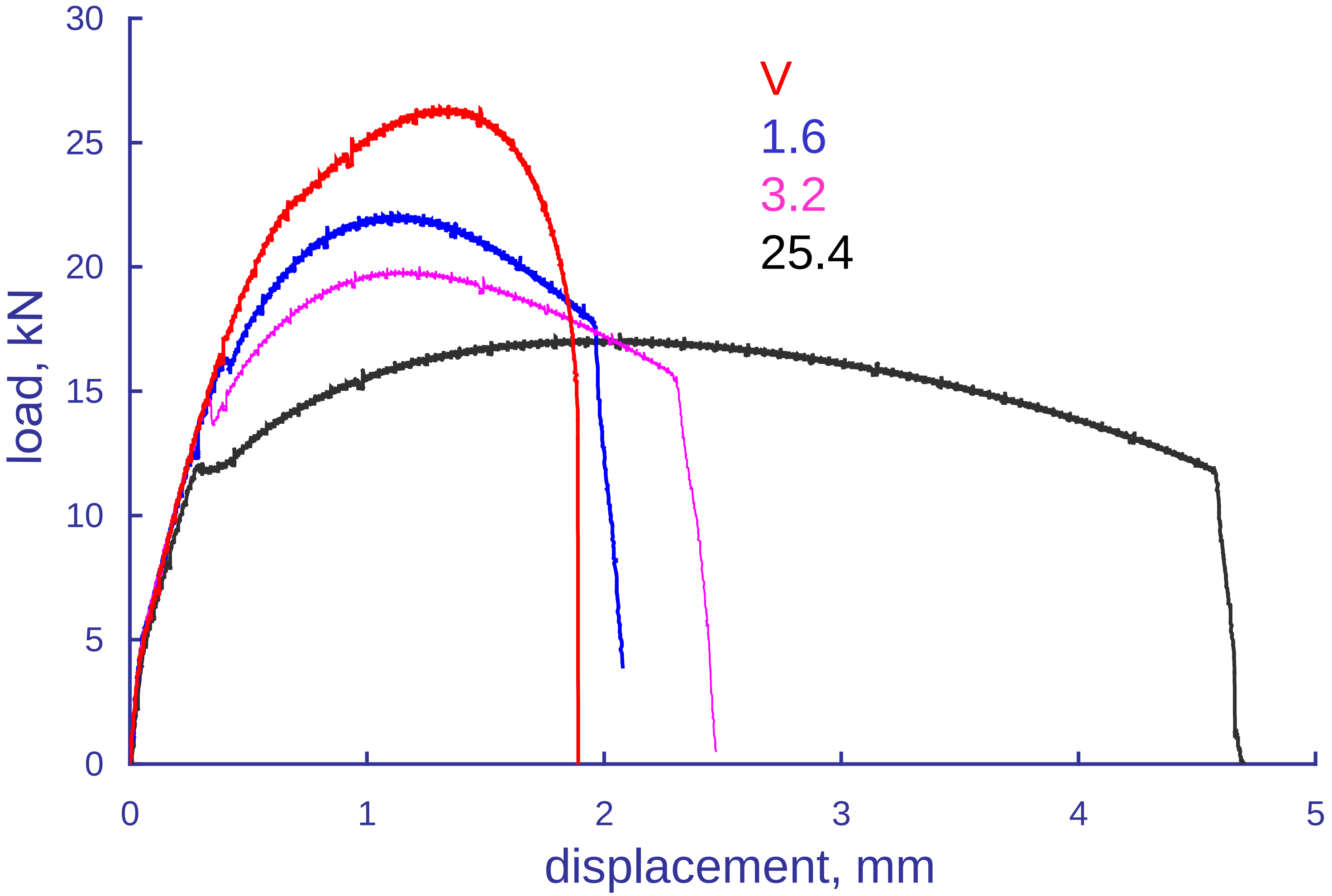


Higher strength and lower ductility

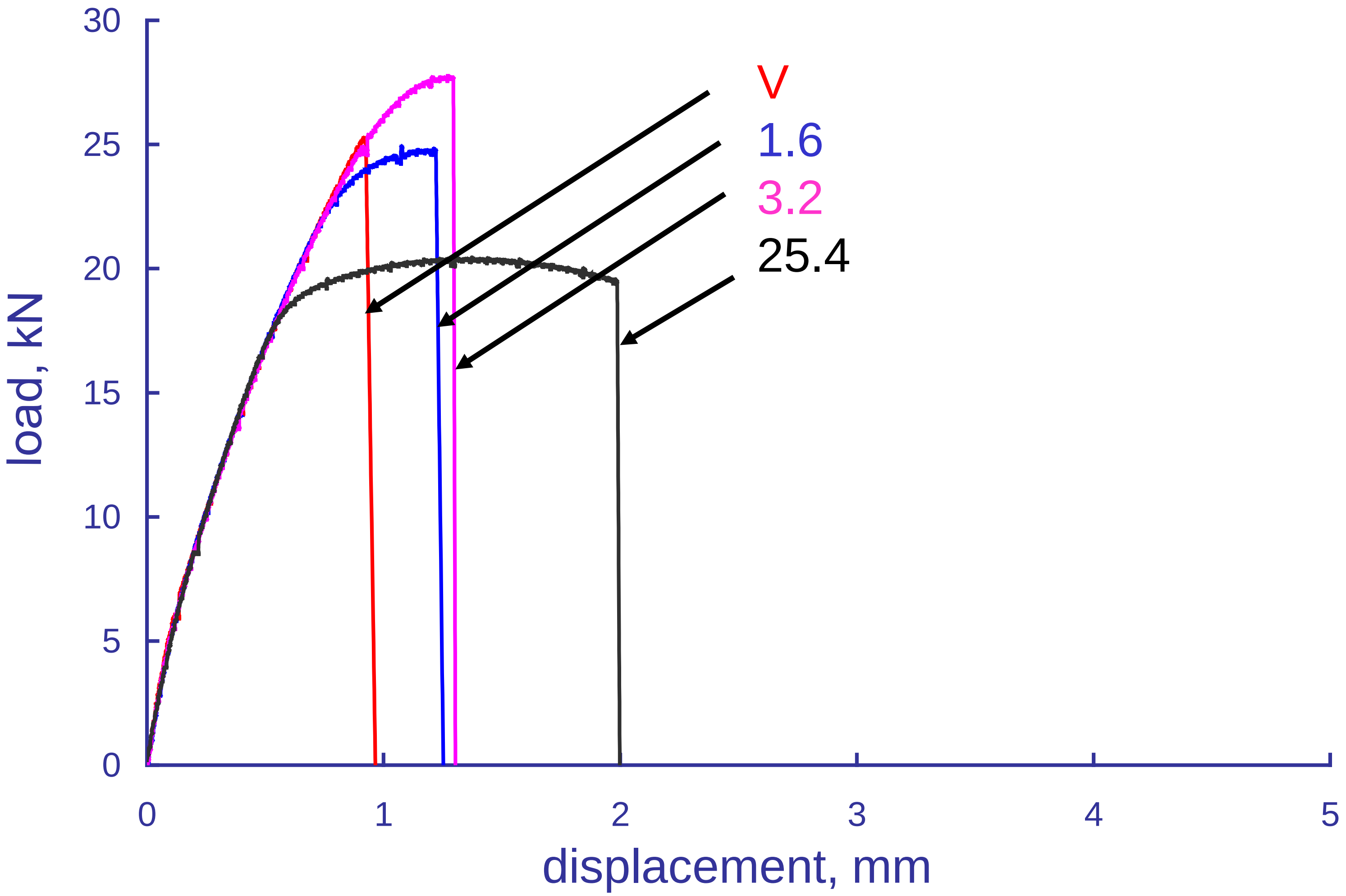
Notched Bars



1018 Steel Test Data



7075-T6 Test Data



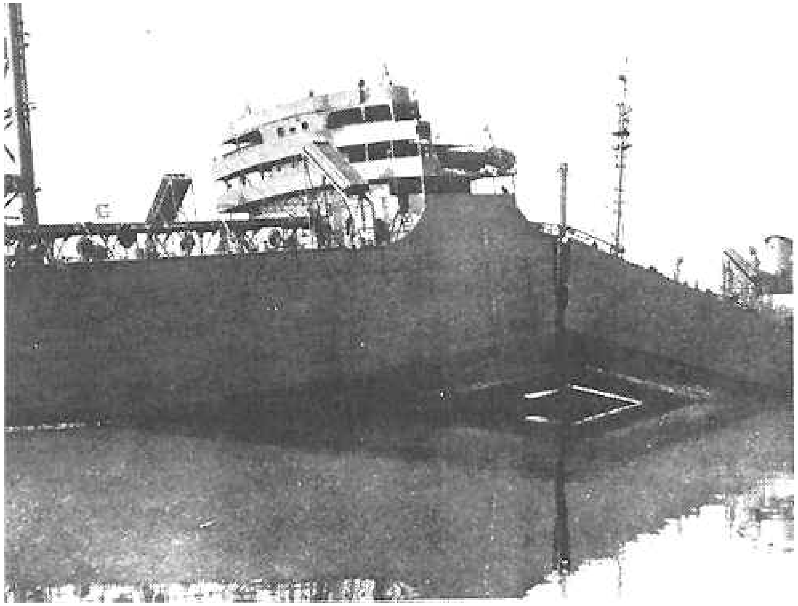
Conclusion

Net section area, state of stress and material strength control the failure load in a structure only in ductile materials. In brittle materials, cracks will form before the maximum load capacity of the structure is reached.

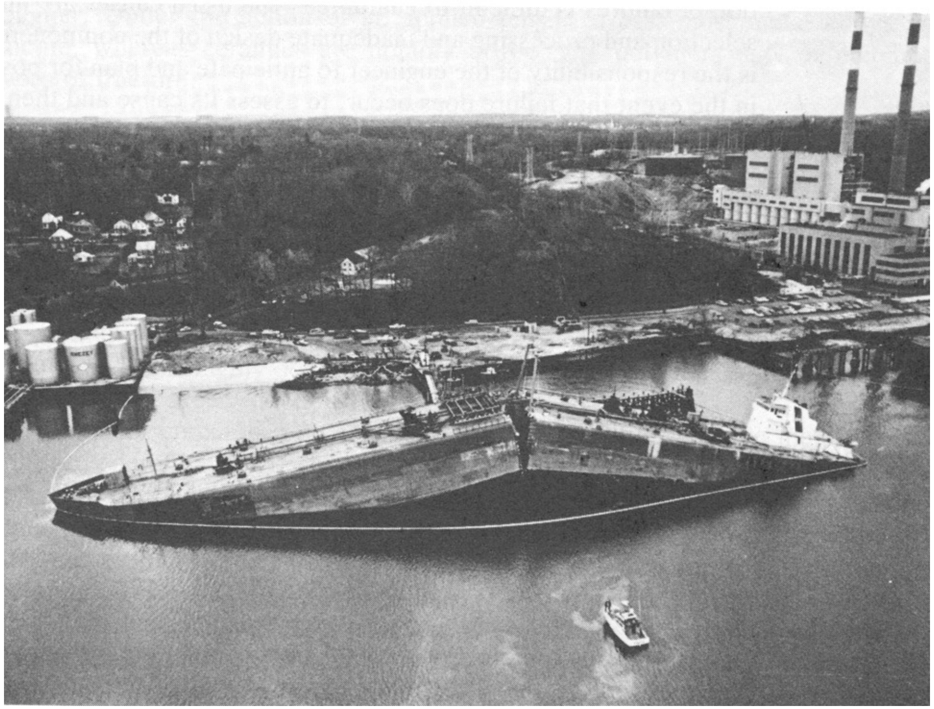
Static Strength and Fracture

- Stress Concentration Factors
- **Fracture Mechanics**
- Approximate Stress Intensity Factors
- Ductile vs. Brittle Fracture

Fractures

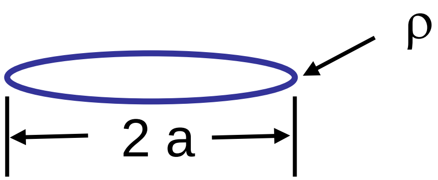


1943



1972

Stress Concentration



$$K_T = 1 + 2\sqrt{\frac{a}{\rho}}$$

for a crack

$a \sim 10^{-3}$

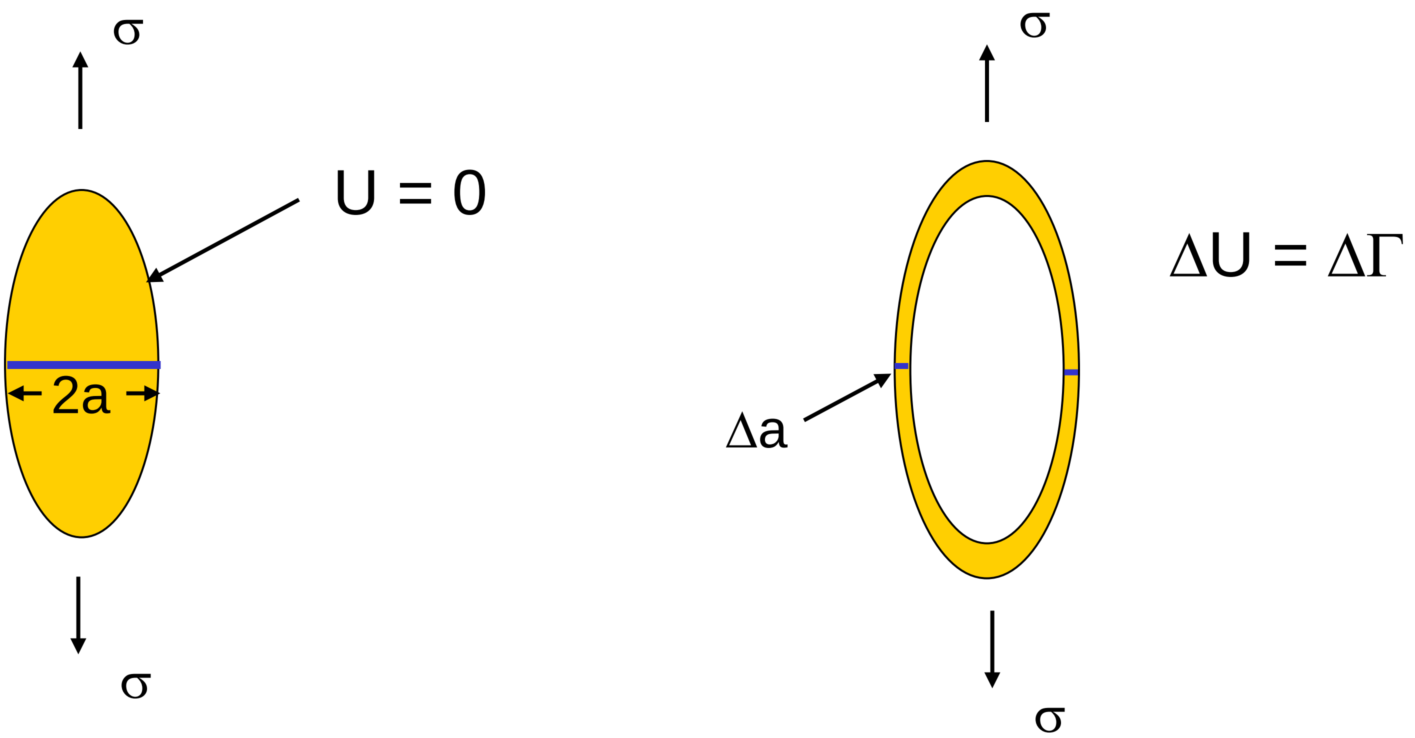
$\rho \sim 10^{-9}$

$K_T \sim 2000$

Fracture Mechanics Parameters

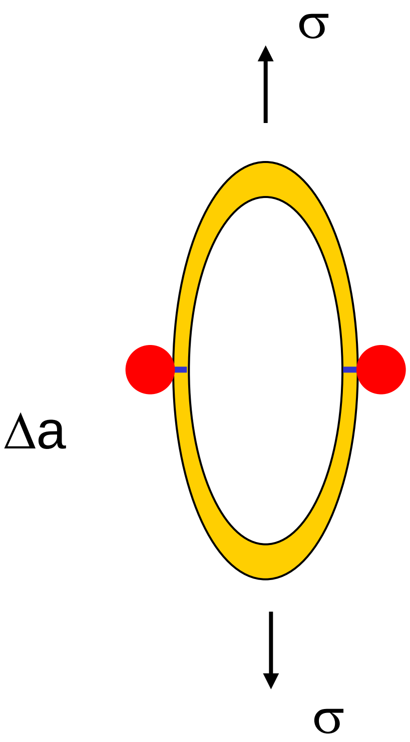
G	strain energy release rate
K	stress intensity factor
J	J-integral
R	crack growth resistance

Strain Energy Release Rate



strain energy ⇒ surface energy

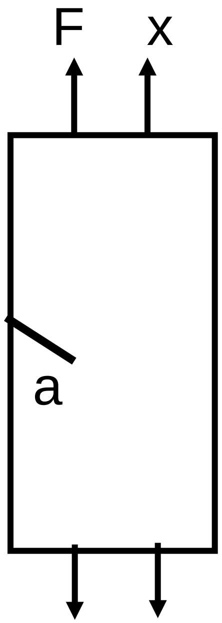
Plastic Energy Term



$$\Delta U = \Delta\Gamma + \Delta P$$

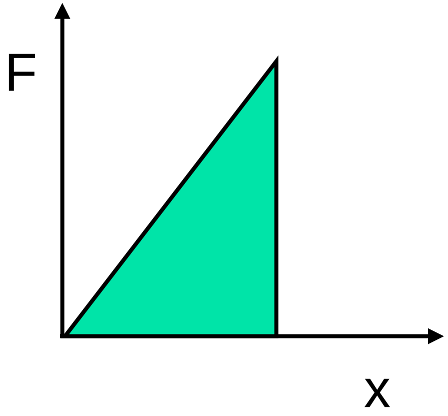
strain energy ⇒ surface energy + plastic energy

Strain Energy Release Rate, G



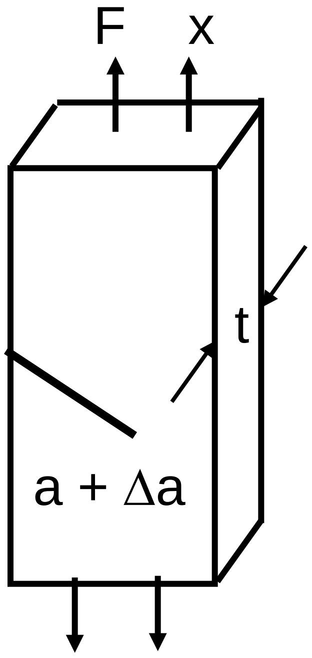
internal strain energy = external work

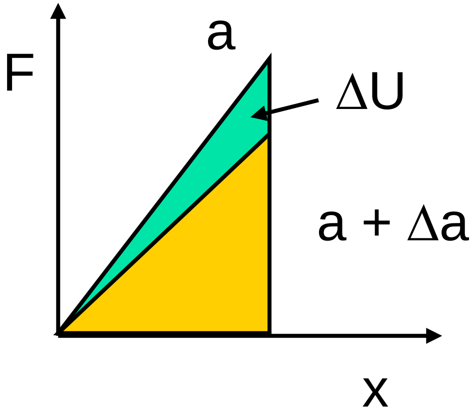
$$U = \frac{1}{2} F x$$



$$G = \frac{\partial U}{\partial a}$$

Strain Energy Release Rate, G

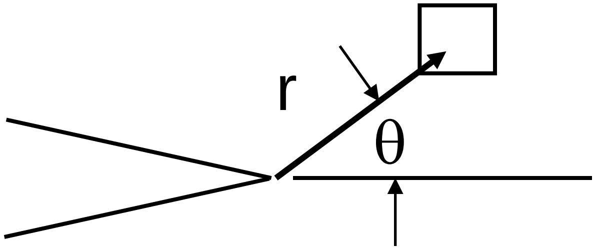




$$G = -\frac{1}{t}\frac{\Delta U}{\Delta a}$$

G is the energy per unit crack area needed to extend a crack

Stress Intensity Factor, K



$$\sigma_x = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_y = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\tau_{xy} = \frac{K}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\varepsilon_y = \frac{\partial v}{\partial x}$$

$$u = \frac{K}{2G}\sqrt{\frac{r}{2\pi}} \cos\frac{\theta}{2}\left[\frac{3-\nu}{1+\nu} - 1 + 2\sin^2\frac{\theta}{2}\right]$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$v = \frac{K}{2G}\sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2}\left[\frac{3-\nu}{1+\nu} - 1 + 2\cos^2\frac{\theta}{2}\right]$$

Design Philosophy

Stress < Strength

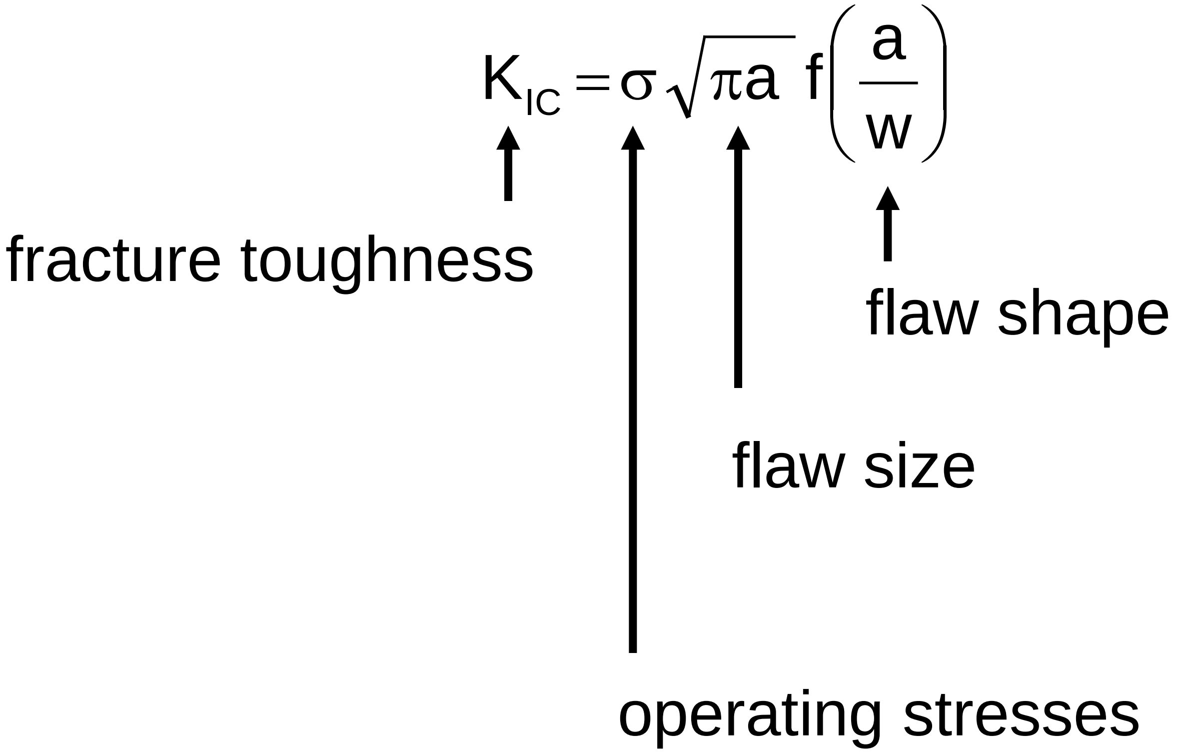
$\sigma < \sigma_y$

Stress Intensity < Fracture Toughness

$K < K_{Ic}$

Two cracks with the same K will have the same behavior

Fracture Toughness



$$K_{IC} = \sigma \sqrt{\pi a}\, f\left(\frac{a}{w}\right)$$

fracture toughness

flaw shape

flaw size

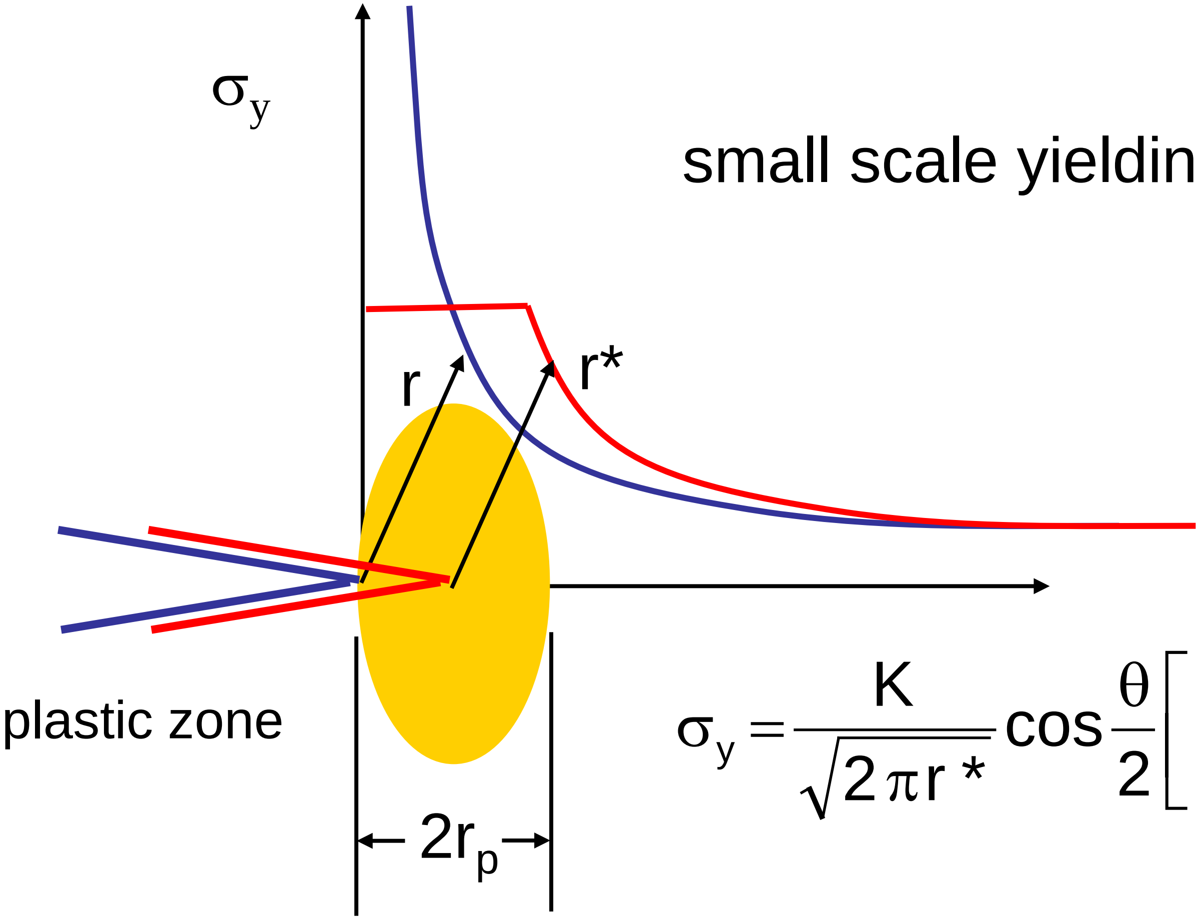
operating stresses

K and G

$$G = \frac{K^2}{E}$$ plane stress

$$G = \frac{\left(1 - \nu^2\right) K^2}{E}$$ plane strain

Elastic-Plastic Stress Field



small scale yielding

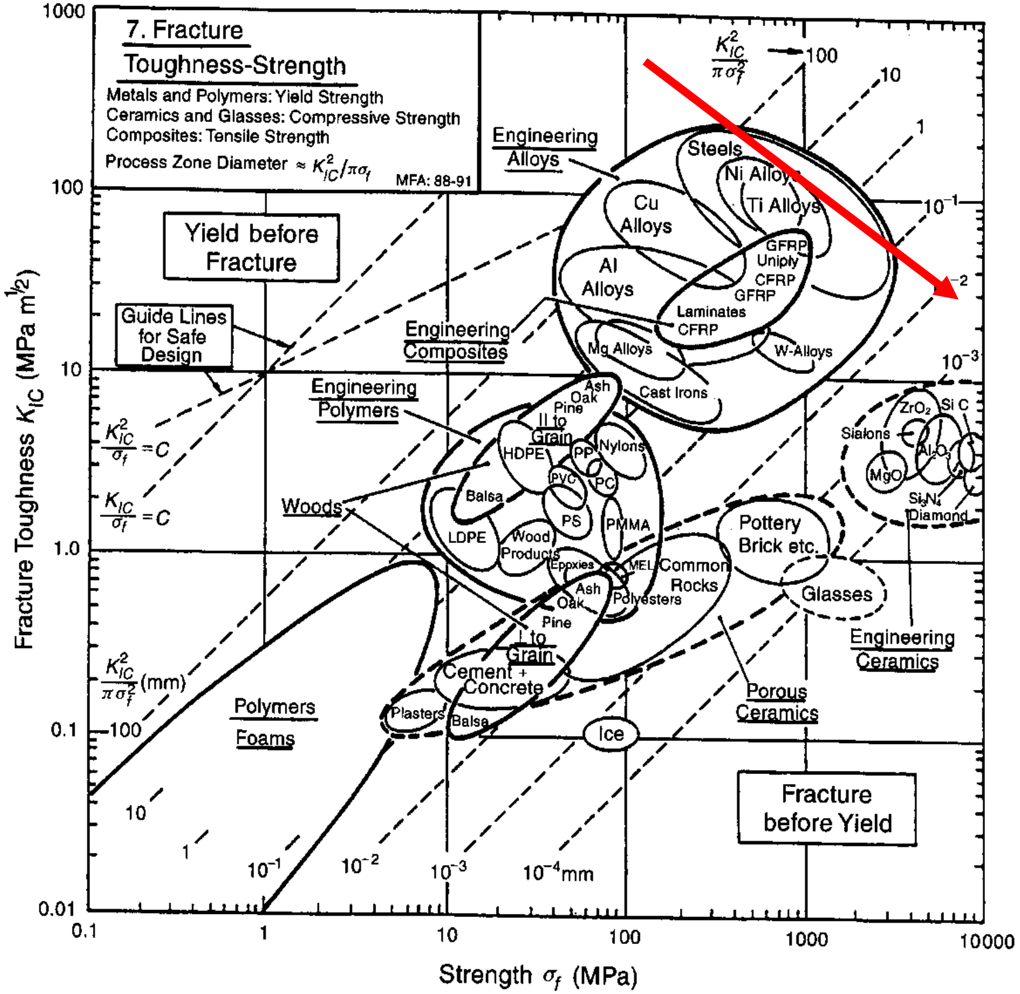
$$\sigma_y = \frac{K}{\sqrt{2\pi r^*}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

Critical Crack Sizes

What would the critical crack size be in a standard tensile test ?

$$a_{critical} = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_f}\right)^2$$

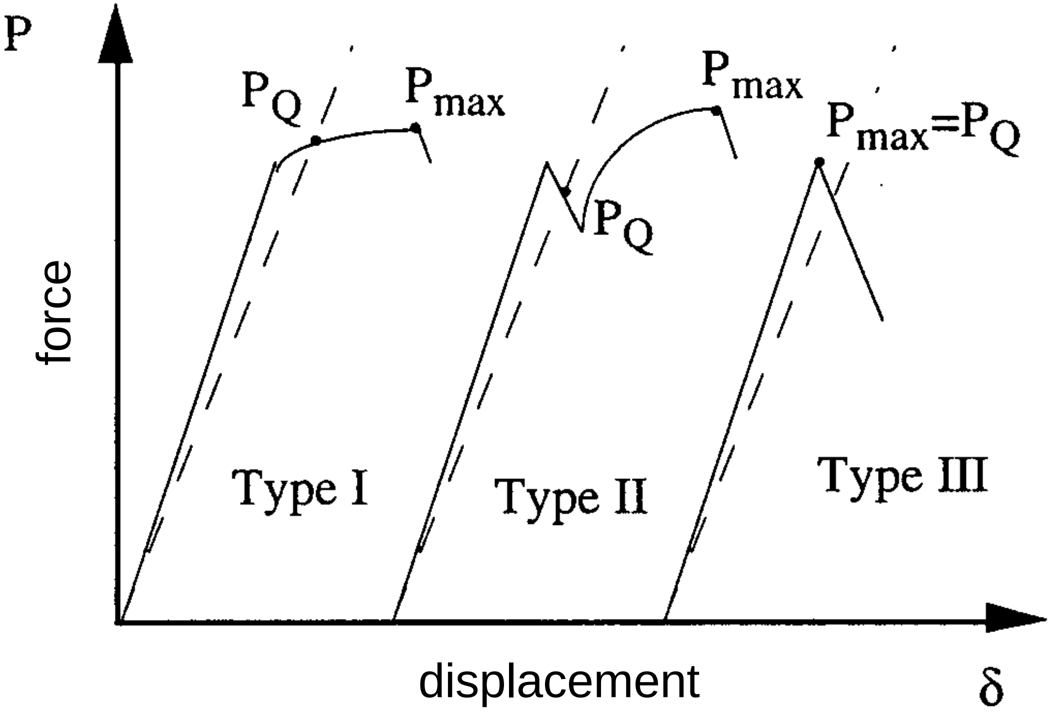
Fracture Toughness



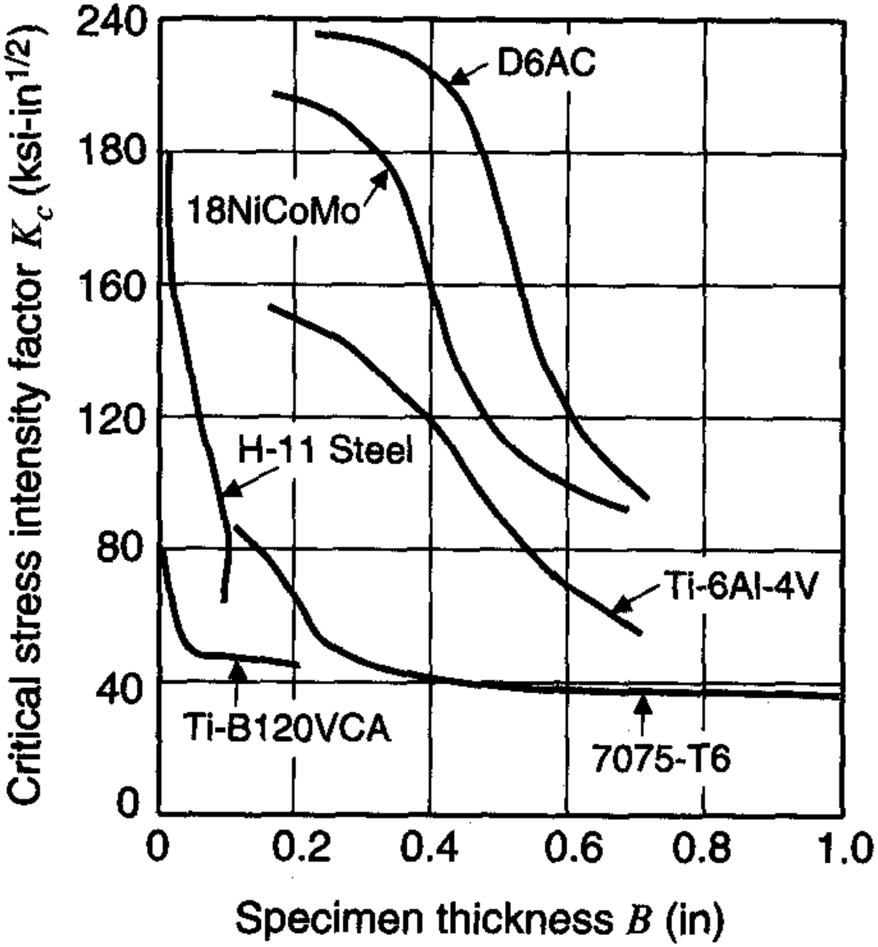
From M F Ashby, Materials Selection in Mechanical Design, 1999, pg 431

Measuring Fracture Toughness

www.enduratec.com

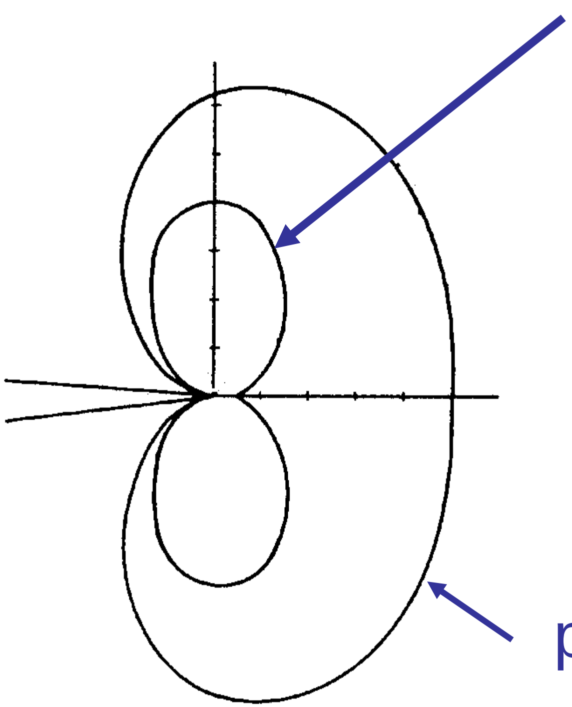


Thickness Effects



From Wilhem “Fracture Mechanics Guidelines for Aircraft Structural Applications” AFFDL-TR-69-111

Plastic Zone Size



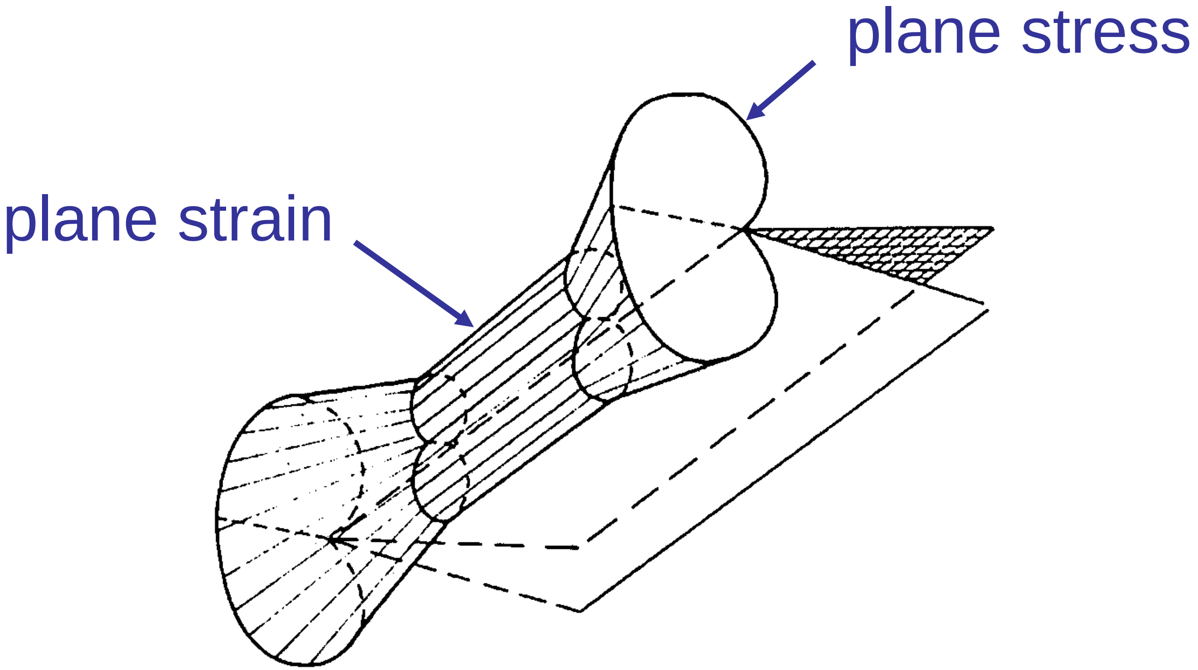
plane strain

$$r_p = \frac{1}{6\pi}\left(\frac{K}{\sigma_{ys}}\right)^2$$

plane stress

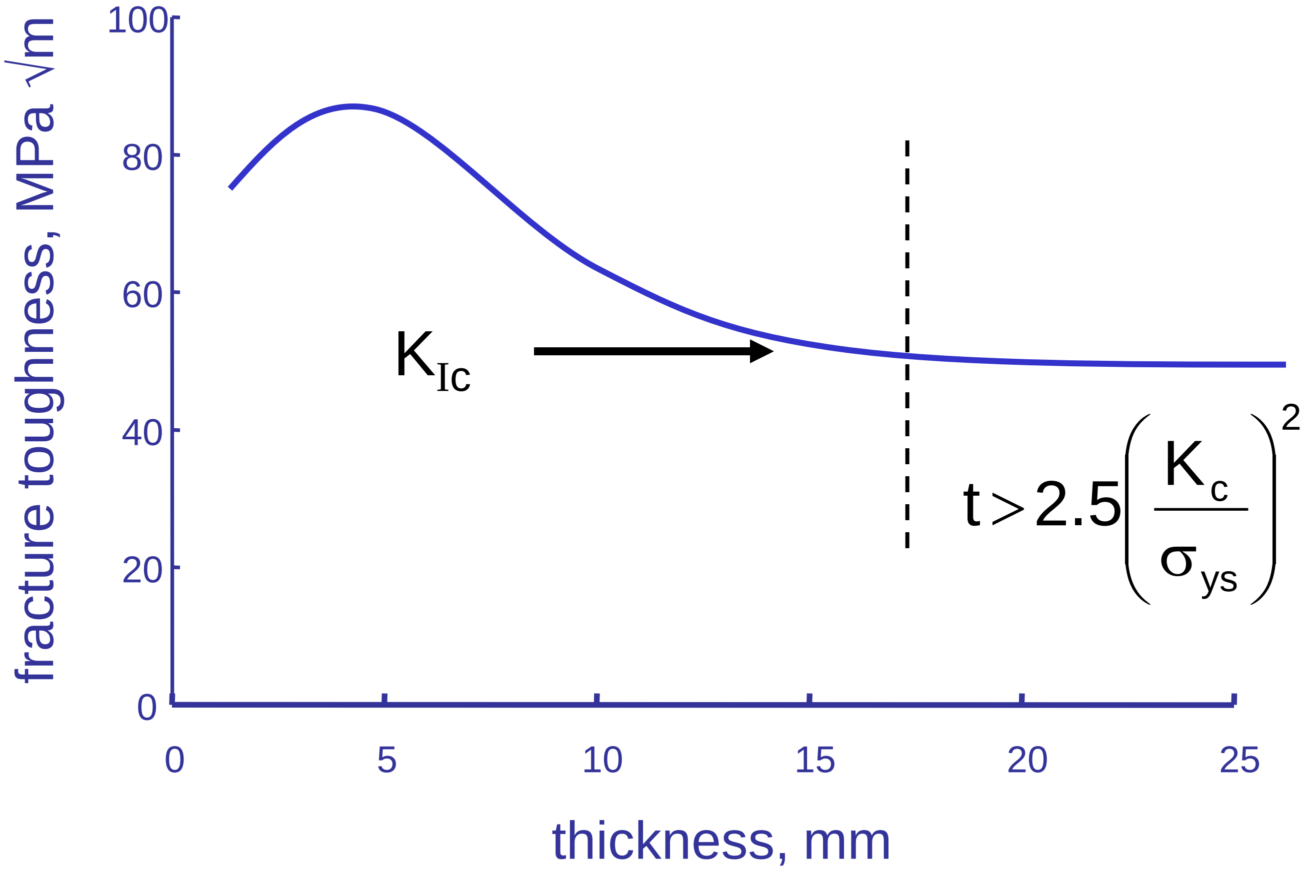
$$r_p = \frac{1}{2\pi}\left(\frac{K}{\sigma_{ys}}\right)^2$$

3D Plastic Zone



Fracture Surfaces

Thickness Requirements



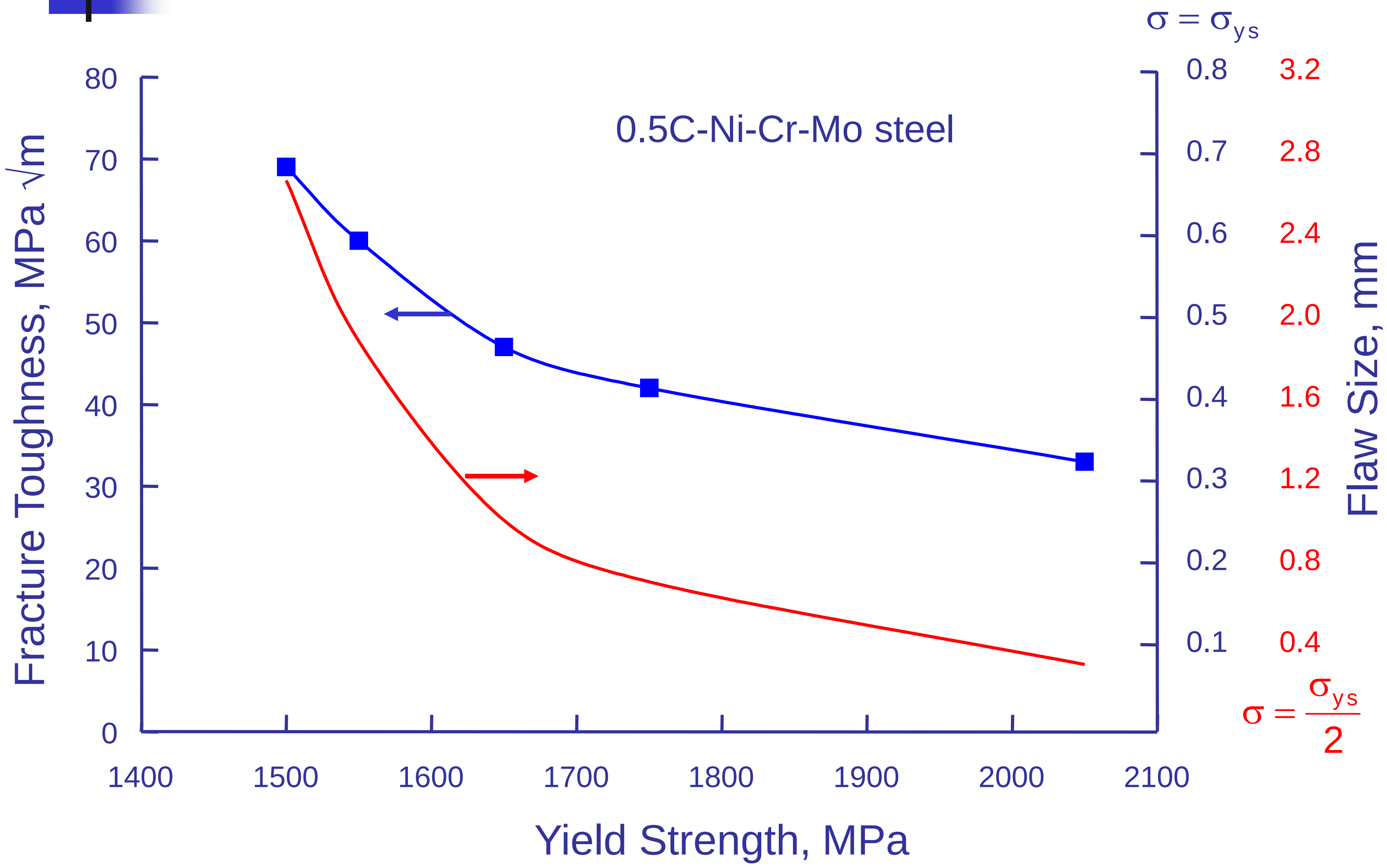
$$t > 2.5\left(\frac{K_c}{\sigma_{ys}}\right)^2$$

Size Requirements

$$t, W-a, a \geq 2.5\left(\frac{K}{\sigma_{ys}}\right)^2 \approx 50 r_p$$

	σ_{ys}	K_{Ic}	t, mm
2024- T3	345	44	40.7
7075 -T6	495	25	6.4
Ti-6Al-4V	910	105	33.3
Ti-6Al-4V	1035	55	7.1
4340	860	99	33.1
4340	1510	60	3.9
17-7 PH	1435	77	7.2
52100	2070	14	0.1

Strength, Toughness, Flaw Size



Silver Bridge

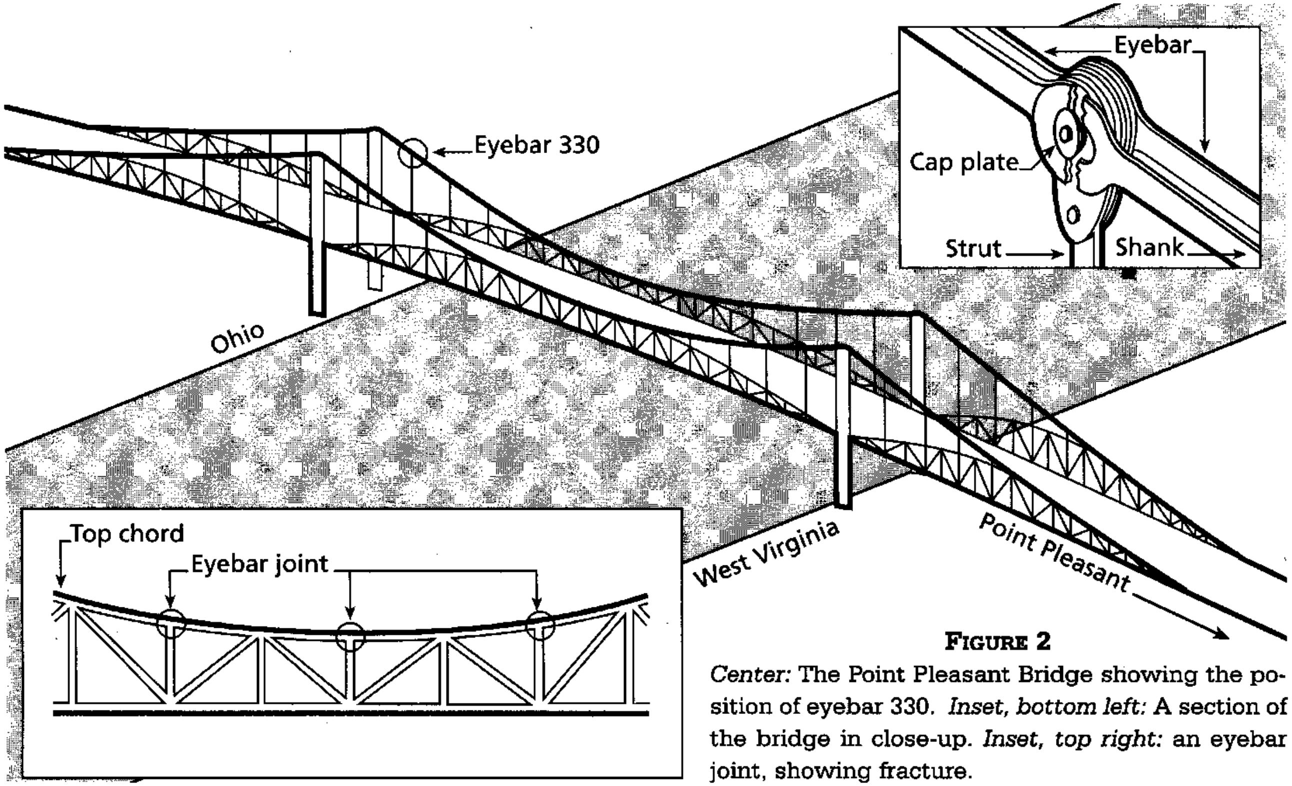
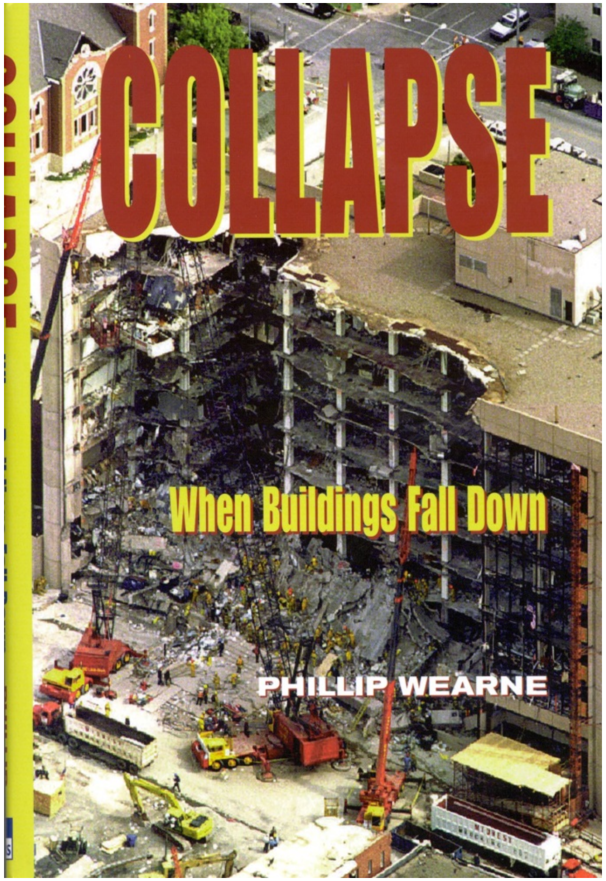


Figure 2

Center: The Point Pleasant Bridge showing the position of eyebar 330. *Inset, bottom left:* A section of the bridge in close-up. *Inset, top right:* an eyebar joint, showing fracture.

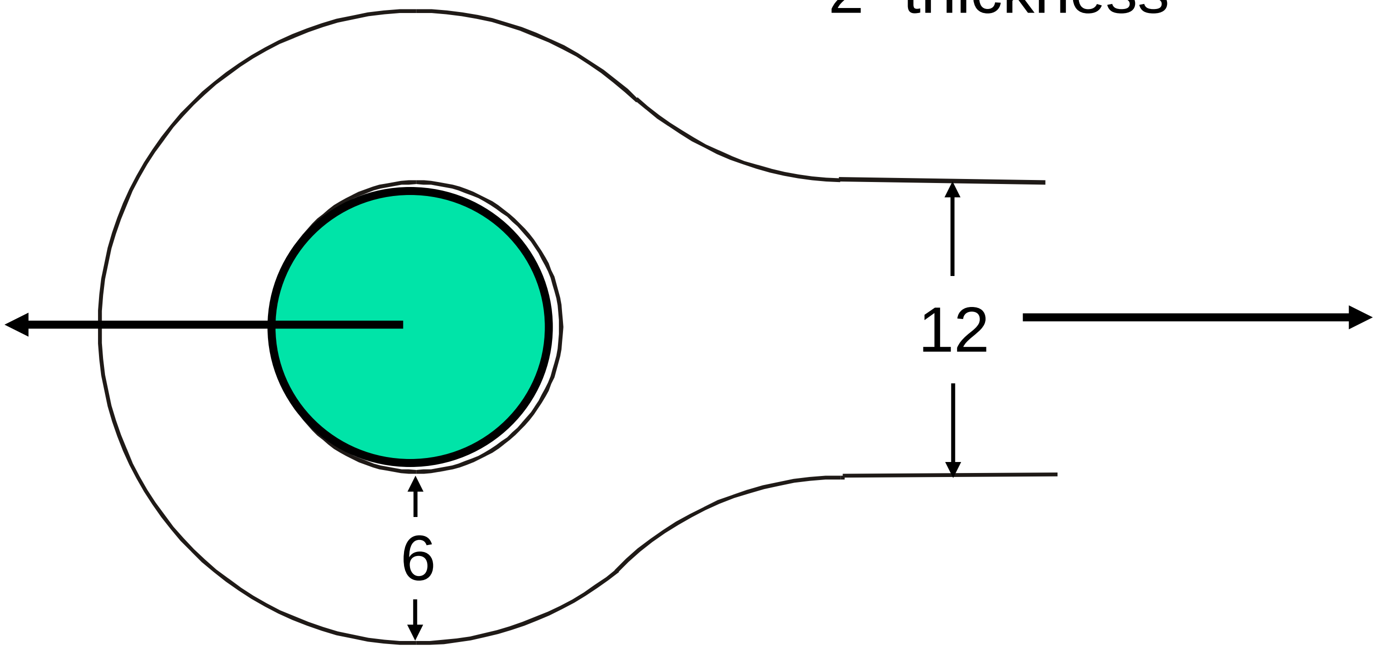
Collapse, Wearne, P. TV Books, NY 1999

Source


COLLAPSE
When Buildings Fall Down
PHILLIP WEARNE
COLLAPSE

Eyebar

12” wide
2” thickness



Material Properties

σ_u = 100 ksi
σ_y = 75 ksi

Working stress 50 ksi

CVN = 2.6 ft-lb at 32° F

CVN = 8.6 ft-lb at 165° F

Fracture Toughness

Barsom-Rolfe $\frac{K_{IC}^{2}}{E} = 2(CVN)^{\frac{3}{2}} \quad (psi-in, ft-lb)$

Corten-Sailors $K_{IC} = 15.5\sqrt{CVN} \quad (ft-lb)$

Roberts-Newton $K_{IC} = 9.35\,CVN^{1.65} \quad (ft-lb)$

Barsom-Rolfe $K_{IC} = 15.9\,ksi\sqrt{in}$

Corten-Sailors $K_{IC} = 25.0\,ksi\sqrt{in}$

Roberts-Newton $K_{IC} = 45.2\,ksi\sqrt{in}$

Average 28.7

Critical Crack Size

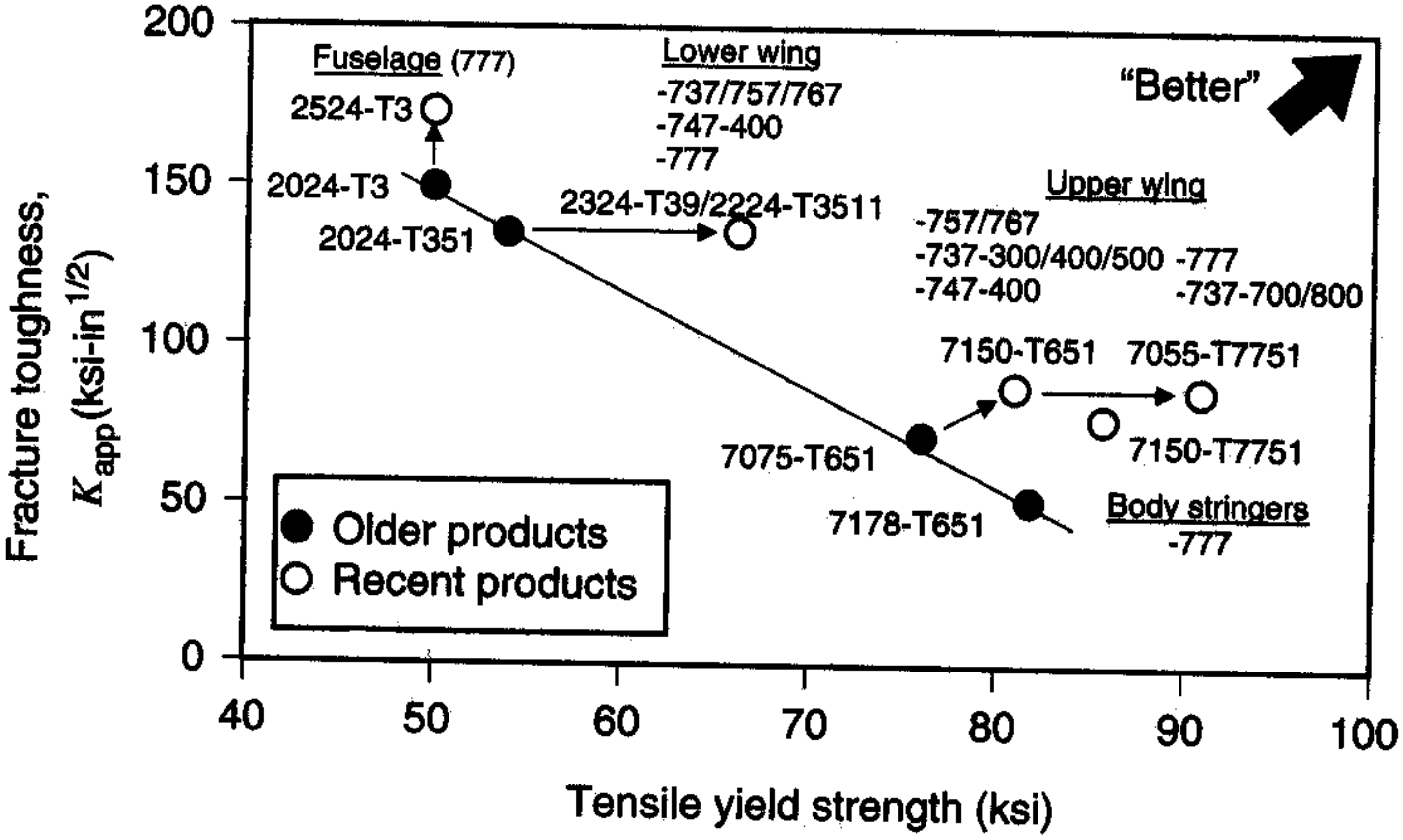
Assume a corner crack

$$K_{IC} = \sigma (1.12)^2 \frac{2}{\pi} \sqrt{\pi a}$$

Let $\sigma = \sigma_y$ a ~ 0.073 inches

Let $\sigma = 50$ a ~ 0.163 inches

Modern Aircraft Materials



Bucci et. al., “Need for New Materials in Aging Aircraft Structures” Journa of Aircraft, Vol. 37, 2000, 122-129

Summary

Both stress and flaw size govern fracture

$$K_{Ic} > \sigma \sqrt{\pi a}\, f\left(\frac{a}{W}\right)$$

Static Strength and Fracture

- Stress Concentration Factors
- Fracture Mechanics
- **Approximate Stress Intensity Factors**
- Ductile vs. Brittle Fracture

Stress Intensity Factors

- Analytical
  - Theory of elasticity
- Numerical
  - Finite element
- Experimental
  - Compliance
- Handbook
- Approximate

Stress Intensity Factors

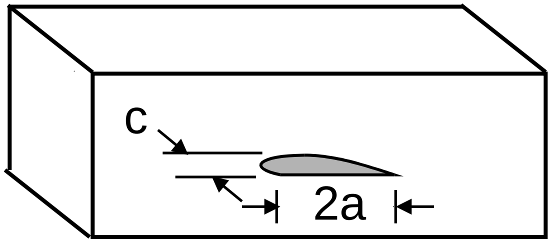
$$K = \frac{M_s M_t}{\Phi} \sigma \sqrt{\pi a}$$

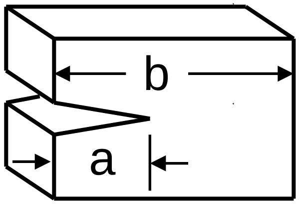
M_s free surface effects

M_t back surface effects

Φ crack shape effects

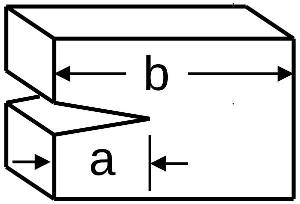
M_s free surface effects



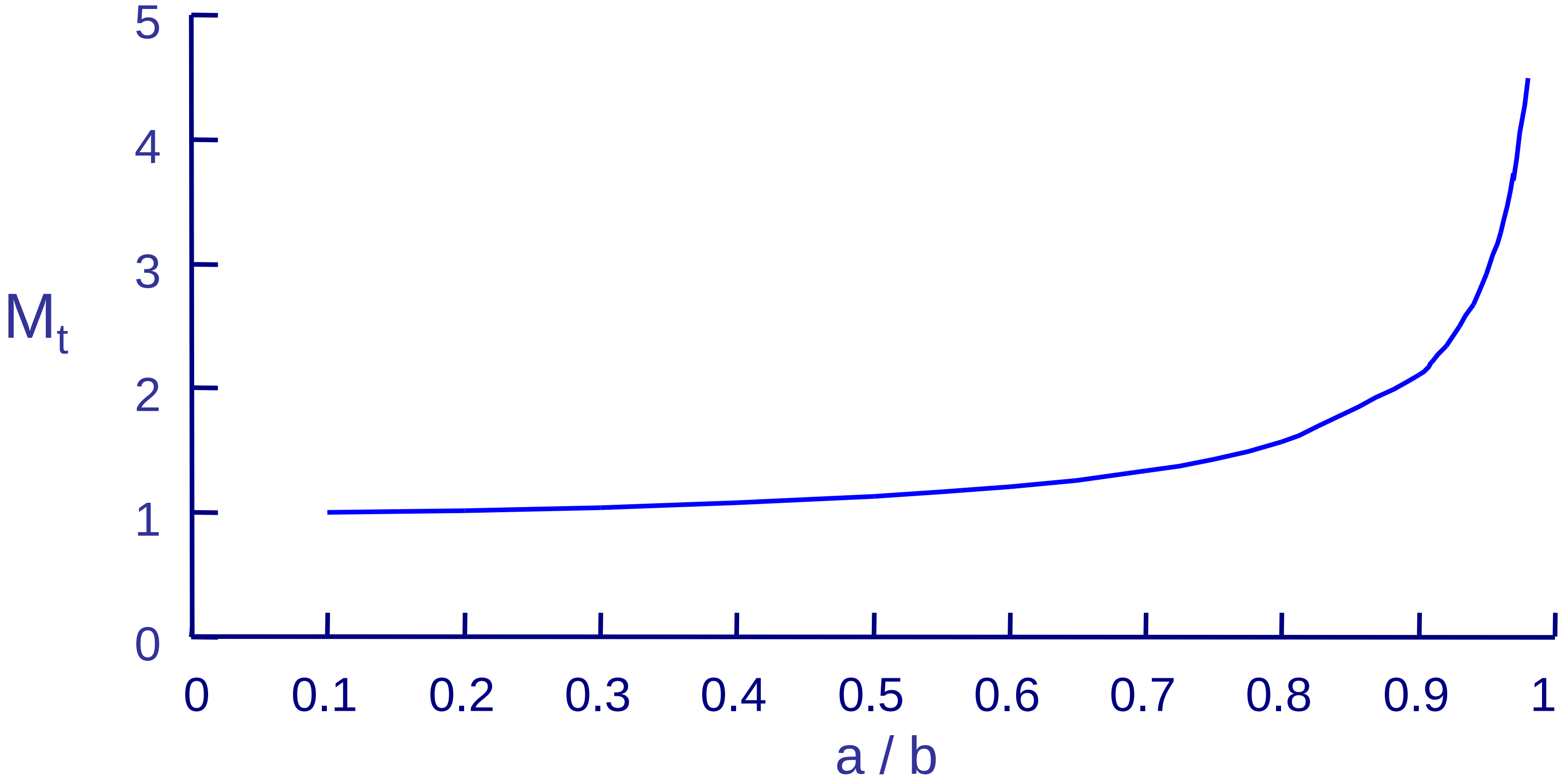


$$M_s = 1.12$$

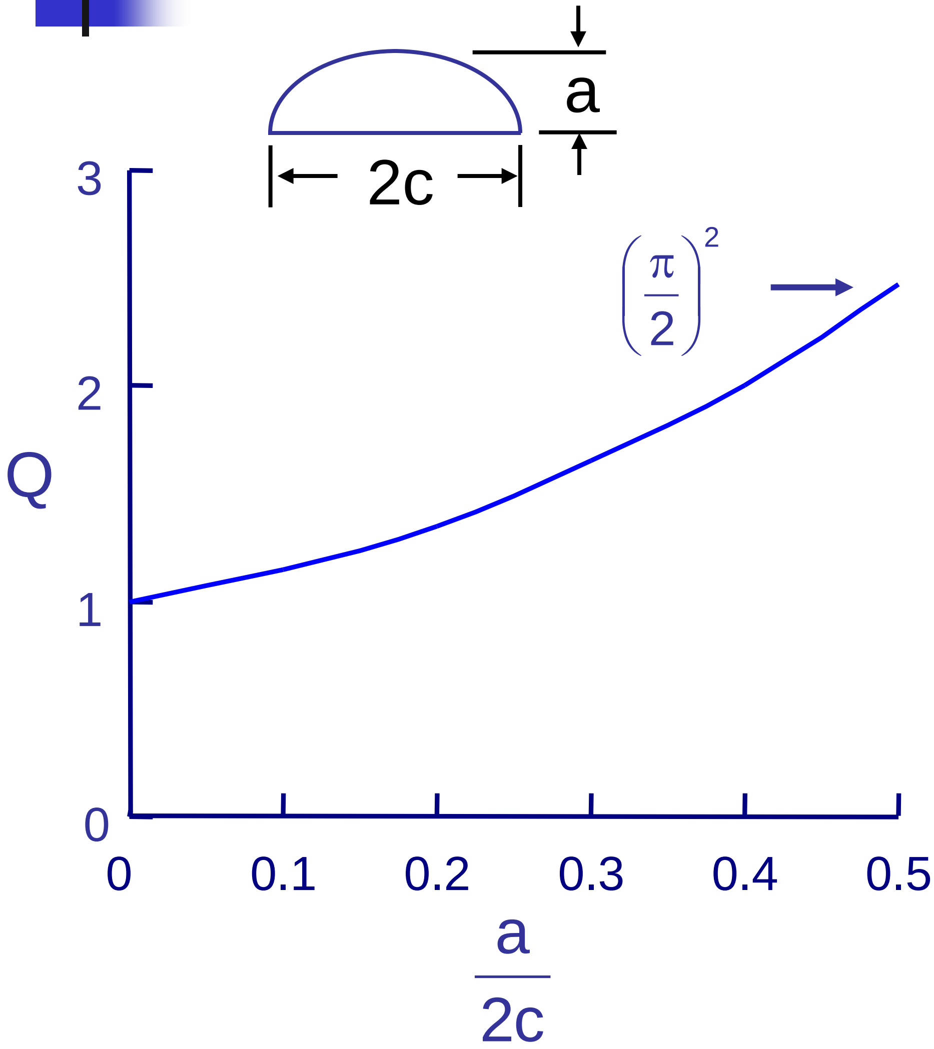
M_t back surface effects



$$M_t = \sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}$$



Φ crack shape effects



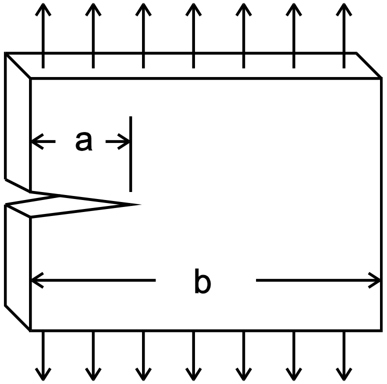
$$\Phi = \sqrt{Q}$$

$$\sqrt{Q} = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \beta}\, d\beta$$

$$k^2 = 1-\left(\frac{a}{c}\right)^2$$

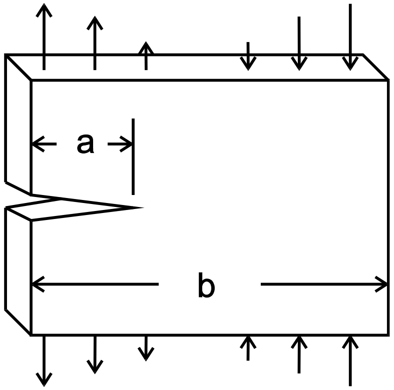
$$Q \cong 1+1.464\left(\frac{a}{c}\right)^{1.65} \qquad a \le c$$

Edge Cracked Plate in Tension



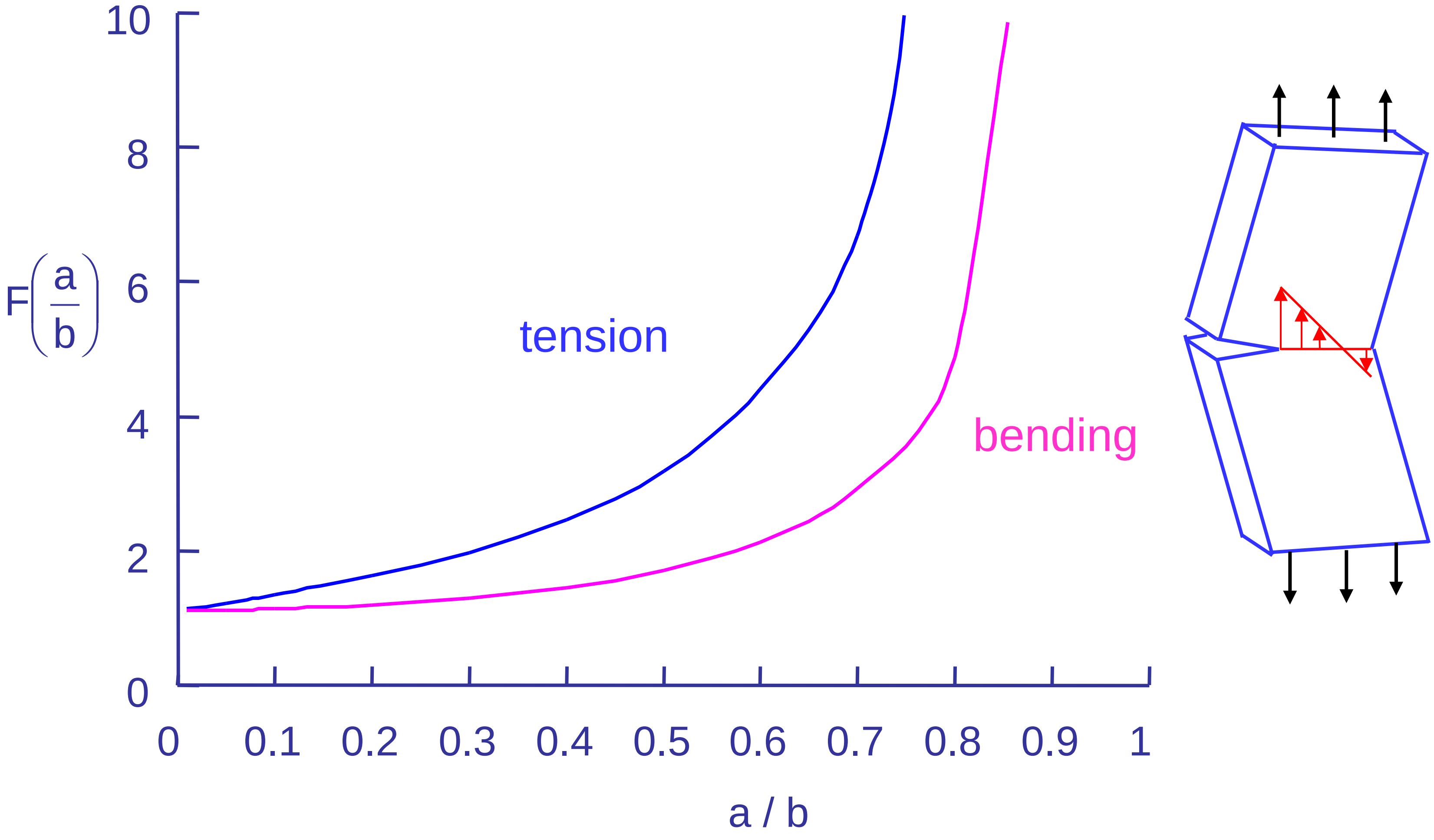
$$F\left(\frac{a}{b}\right)=\sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}\left(\frac{0.752+2.02\frac{a}{b}+0.37(1-\sin\frac{\pi a}{2b})^{3}}{\cos\frac{\pi a}{2b}}\right)$$

Edge Cracked Plate in Bending

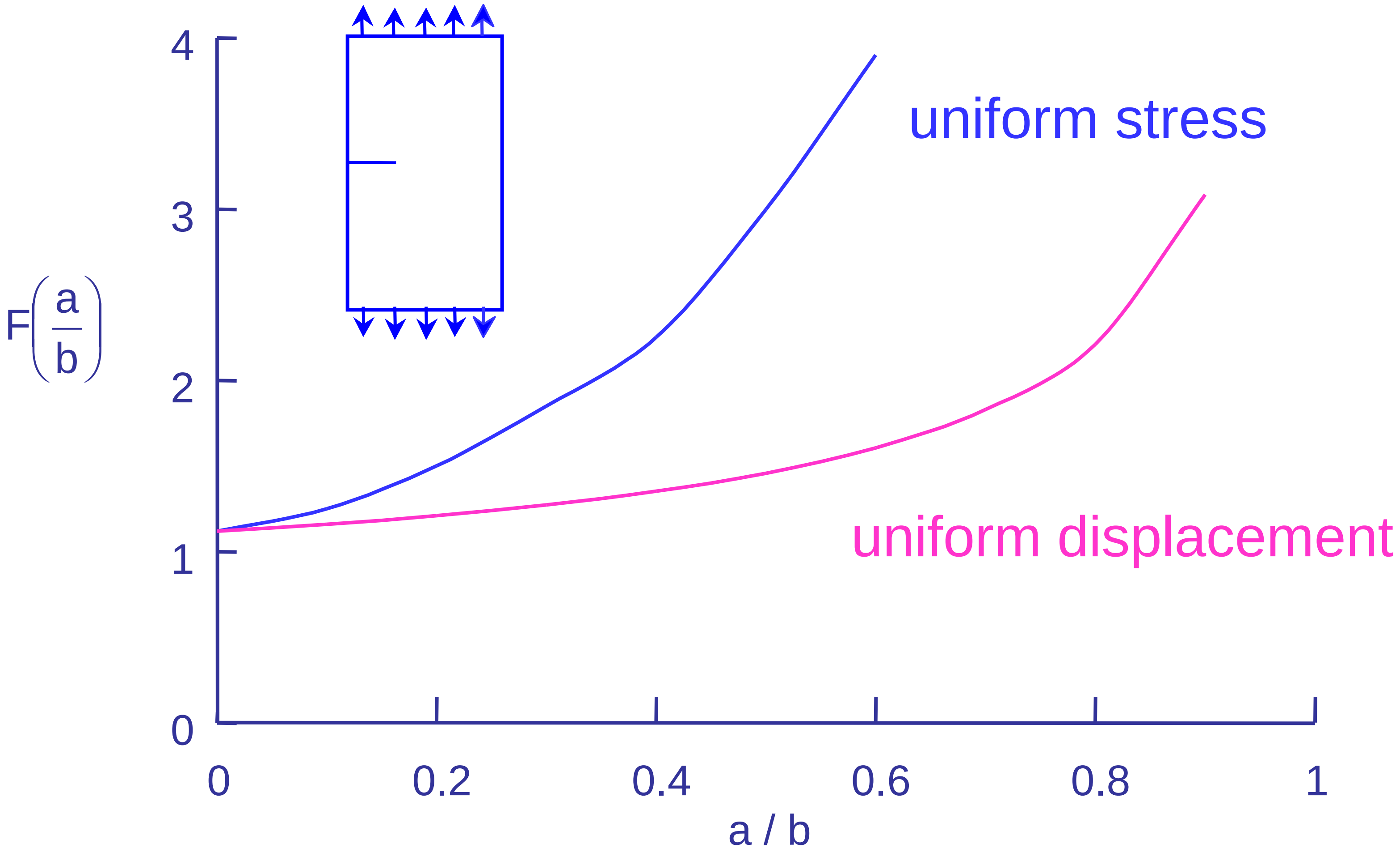


$$F\left(\frac{a}{b}\right)=\sqrt{\frac{2b}{\pi a}\tan\frac{\pi a}{2b}}\left(\frac{0.923+0.199(1-\sin\frac{\pi a}{2b})^4}{\cos\frac{\pi a}{2b}}\right)$$

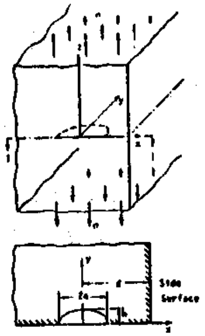
Tension and Bending

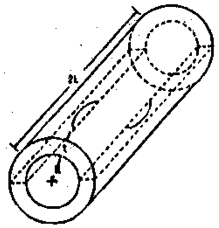


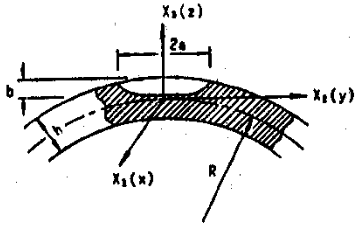
Boundary Conditions



Handbook





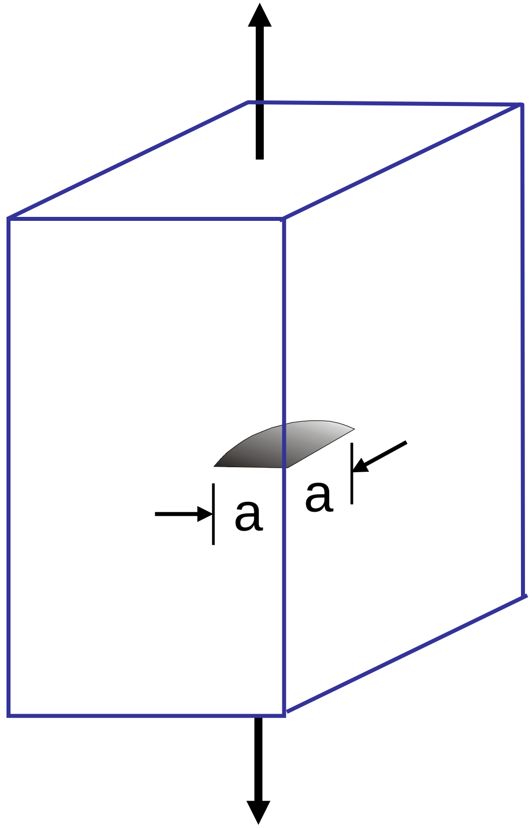


Handbook

Stress Intensity Factors Handbook

Y. Murakami Editor, Pergamon Press

Useful approximations

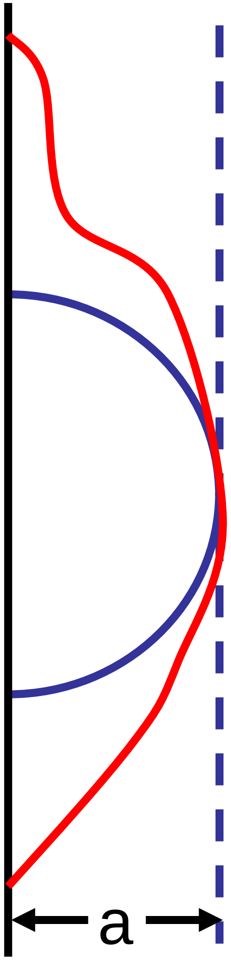


Corner crack

Two free edges
Semicircular shape

$$K = \sigma (1.12)^2 \frac{2}{\pi} \sqrt{\pi a}$$

Crack Shape



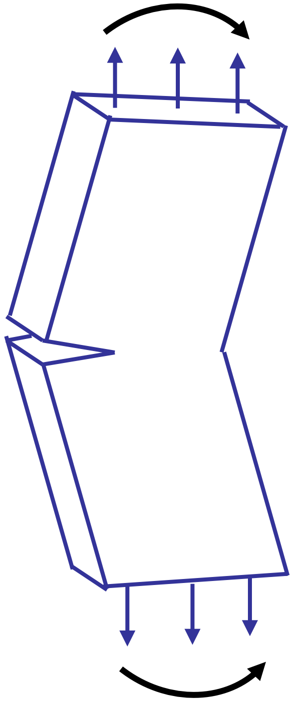
Through crack

$$K = \sigma 1.12 \sqrt{\pi a}$$

Semielliptical crack

$$K = \sigma 1.12 \frac{2}{\pi} \sqrt{\pi a} = \sigma 0.71 \sqrt{\pi a}$$

Superposition



tension + bending

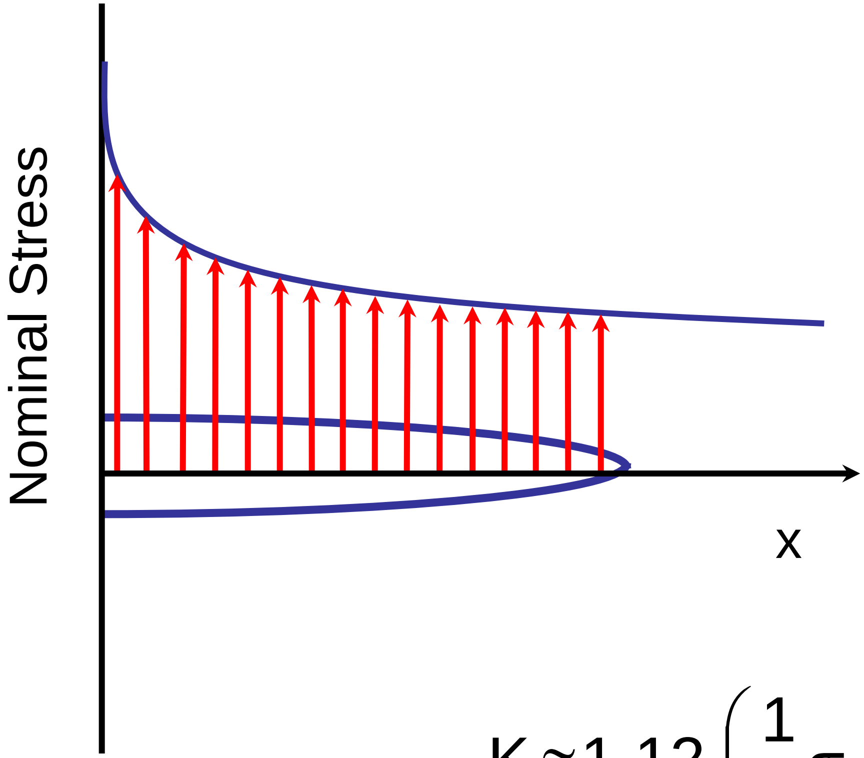
Crack tip stresses:

$$\sigma_{ij} = \frac{K_{tension}}{\sqrt{2\pi r}} f_{ij}(\theta) + \frac{K_{bending}}{\sqrt{2\pi r}} f_{ij}(\theta)$$

$$\sigma_{ij} = \frac{K_{tension} + K_{bending}}{\sqrt{2\pi r}} f_{ij}(\theta) = \frac{K_{total}}{\sqrt{2\pi r}} f_{ij}(\theta)$$

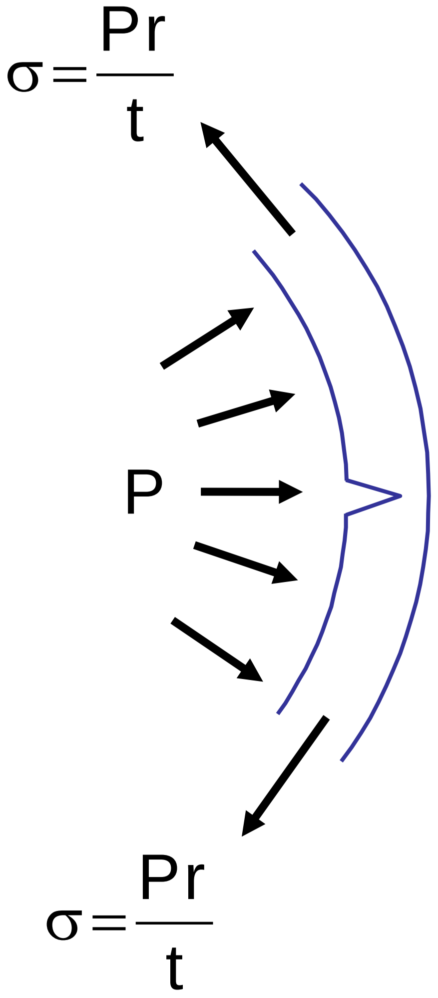
$$K_{total} = K_{tension} + K_{bending}$$

Stress Gradients



$$K \cong 1.12\left(\frac{1}{3}\sigma_{\text{average}}+\frac{2}{3}\sigma_{\text{crack tip}}\right)\sqrt{\pi a}$$

Pressure Vessel



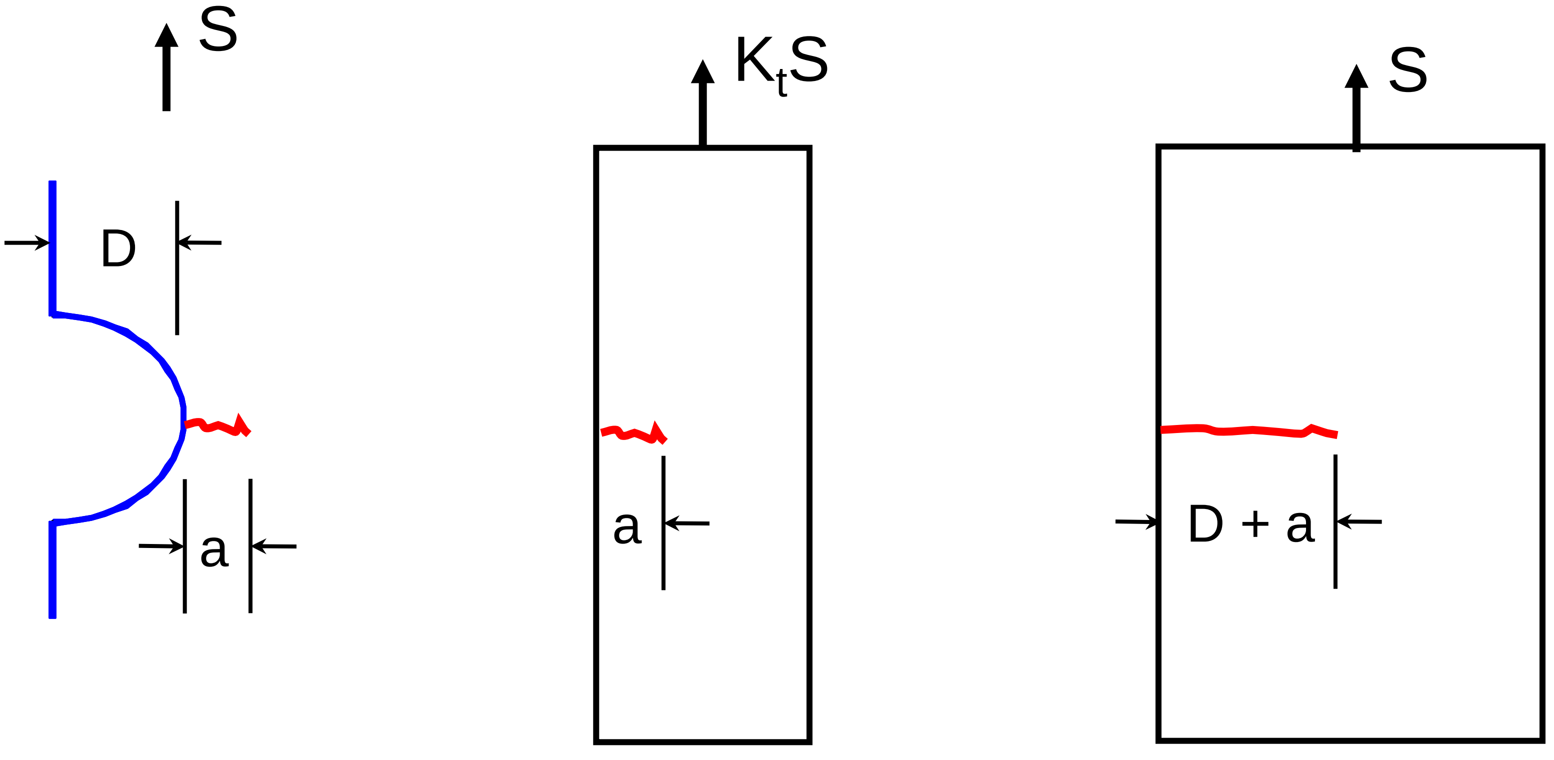
$$\sigma = \frac{Pr}{t}$$

$$\sigma = \frac{Pr}{t}$$

$$K = \frac{Pr}{t}\sqrt{\pi a} + P\sqrt{\pi a}$$

$$K = P\left(\frac{r}{t} + 1\right)\sqrt{\pi a}$$

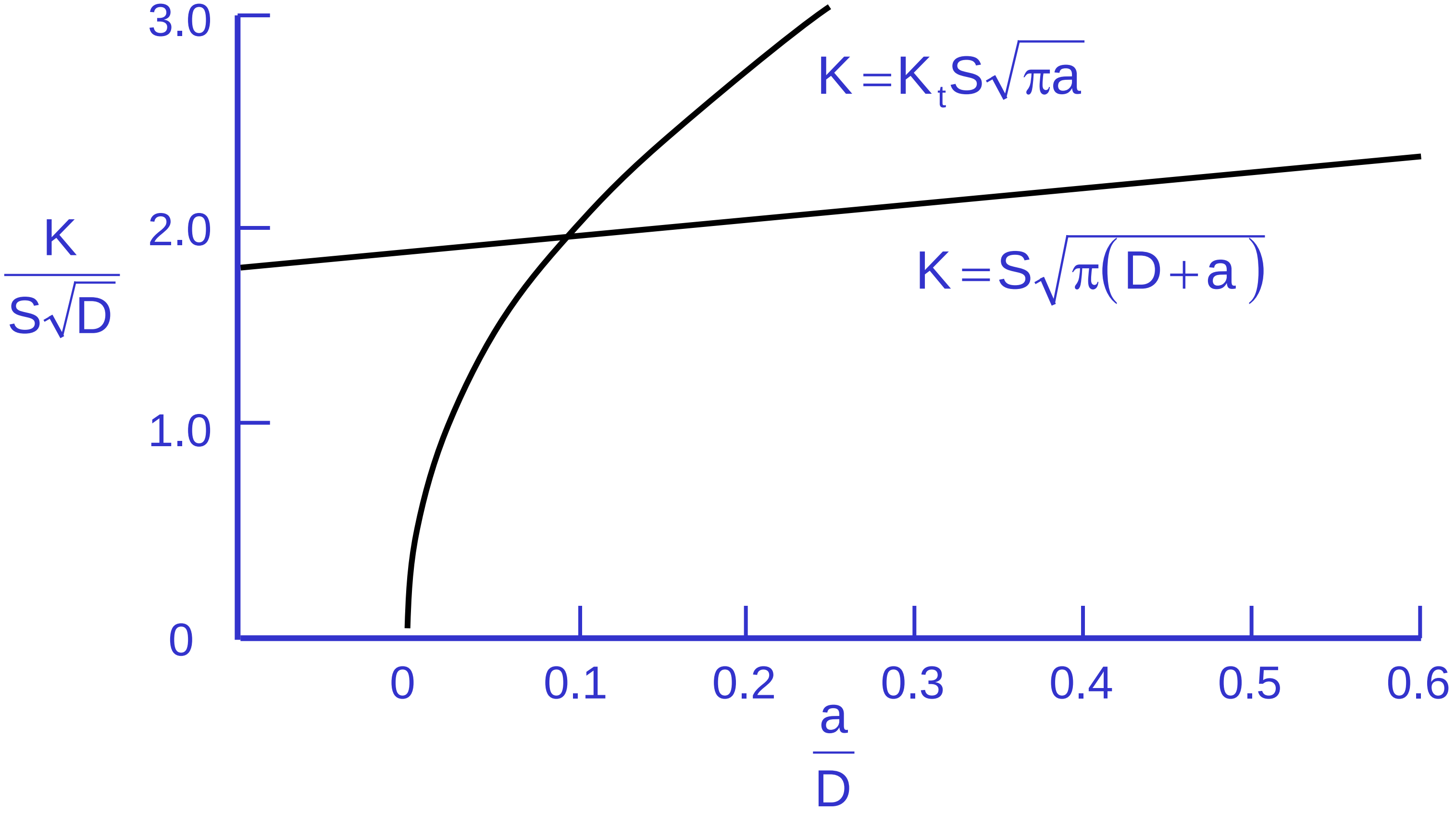
Cracks at Notches



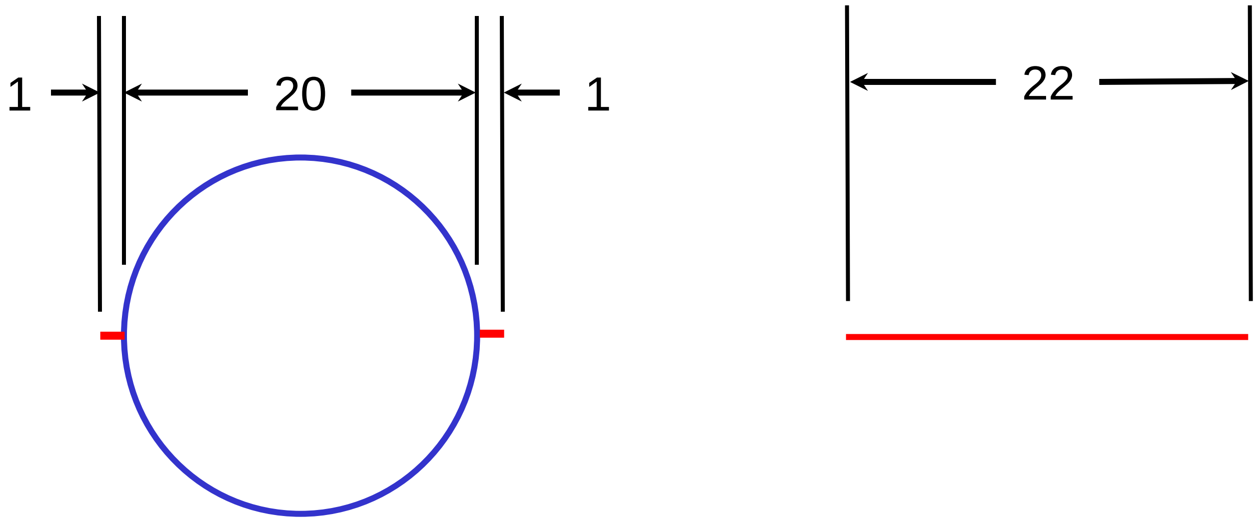
a << D

a >> D

Stress Intensity Factors



Cracks at Holes



Once a crack reaches 10% of the hole radius, it behaves as if the hole was part of the crack

Summary

$$K_{Ic} > \sigma \sqrt{\pi a}\, f\left(\frac{a}{W}\right)$$

$$\sigma \sim \sigma_{ys} / 2$$

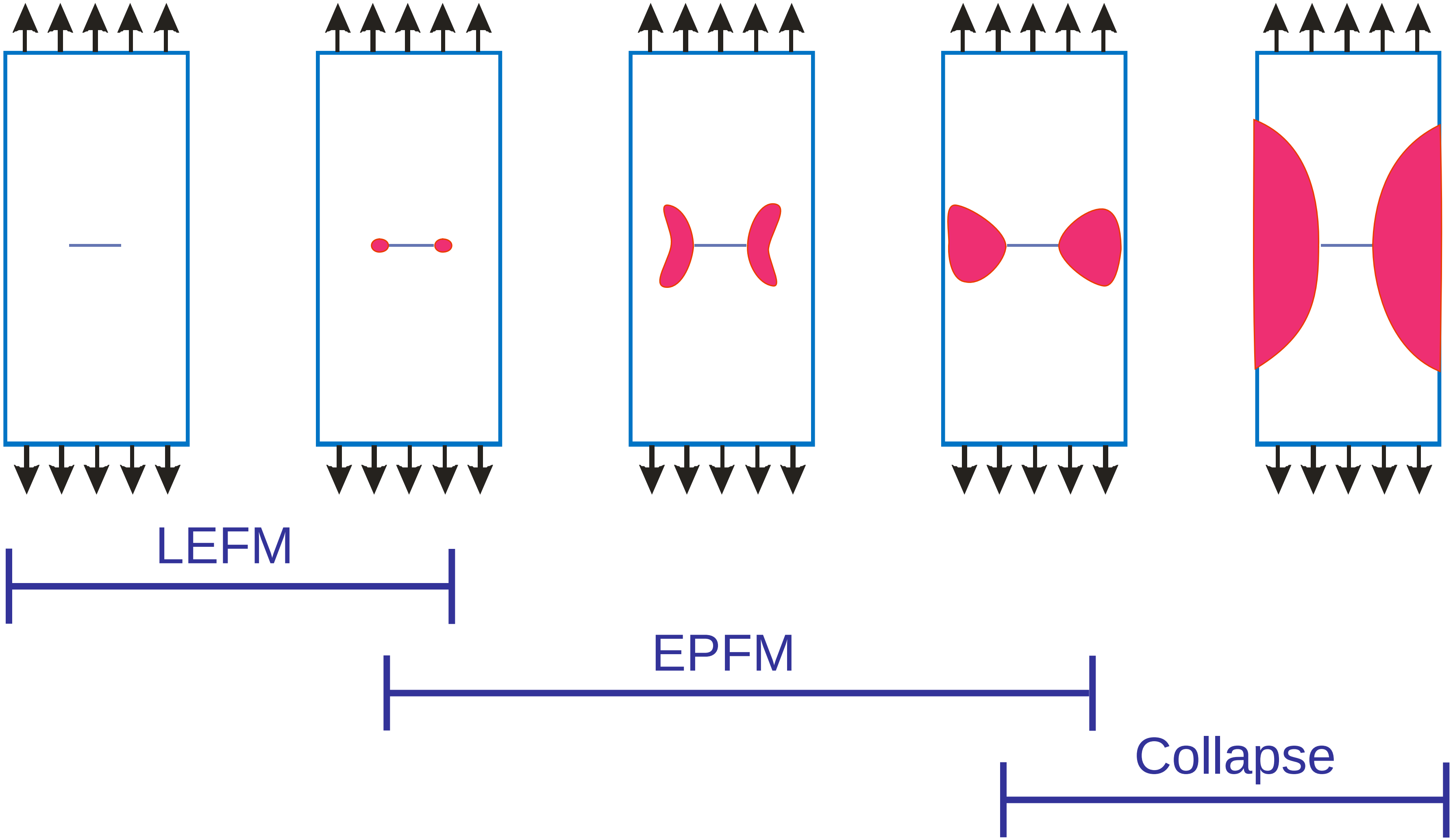
$$a \sim 0.1\ \text{mm} - 100\ \text{mm}$$

$$f\left(\frac{a}{W}\right) \sim 1 - 2$$

Static Strength and Fracture

- Stress Concentration Factors
- Fracture Mechanics
- Approximate Stress Intensity Factors
- **Ductile vs. Brittle Fracture**

Fracture Behavior

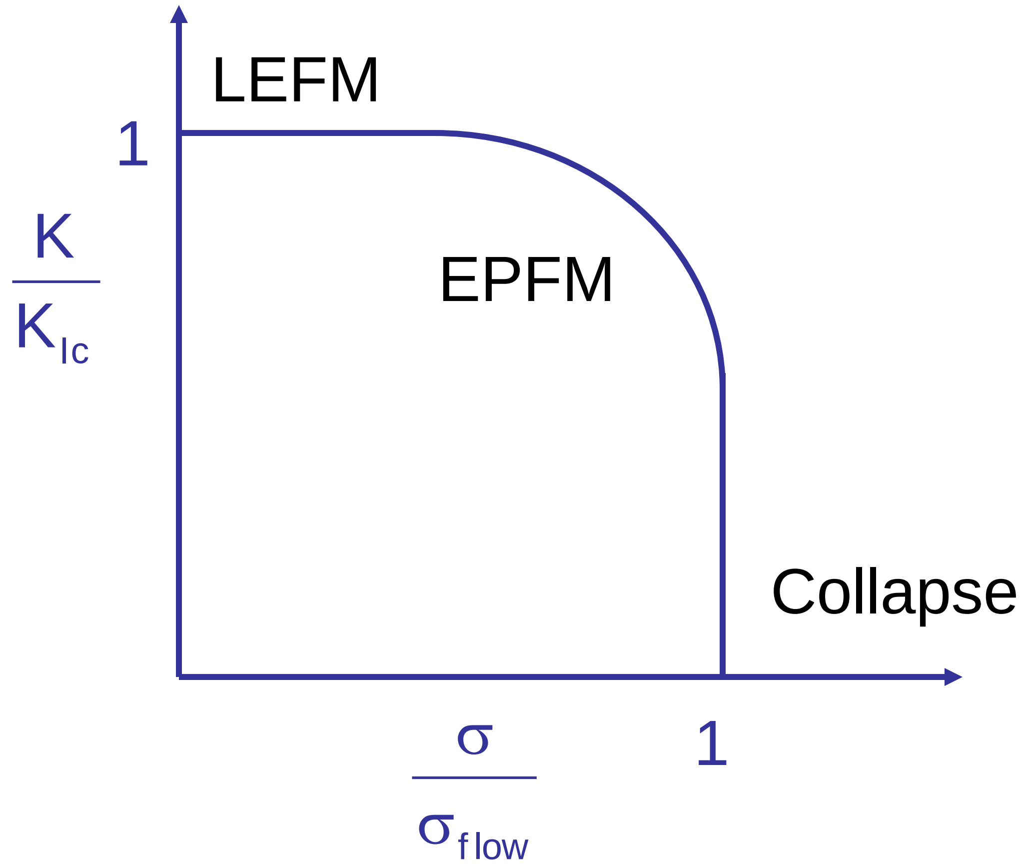


LEFM

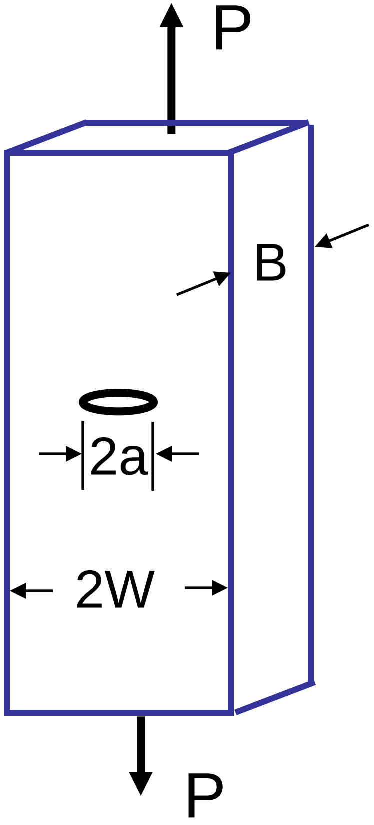
EPFM

Collapse

Failure Analysis Diagram



Plastic Collapse

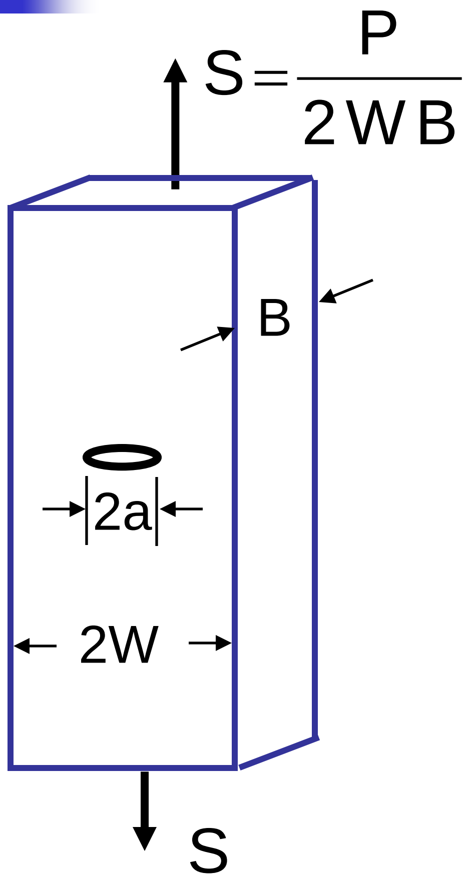


$$\frac{P}{2(W-a)B} < \sigma_{flow}$$

$$S = \frac{P}{2WB}$$ Nominal stress

$$S = \sigma_{flow}\left(1 - \frac{a}{W}\right)$$

Fracture



$$S\sqrt{\pi a}\, f\left(\frac{a}{W}\right) = K_{Ic}$$

$$S = \frac{K_{Ic}}{\sqrt{\pi \frac{a}{W}}\sqrt{W}\, f\left(\frac{a}{W}\right)}$$

Fracture vs. Collapse

Fracture and collapse equally likely

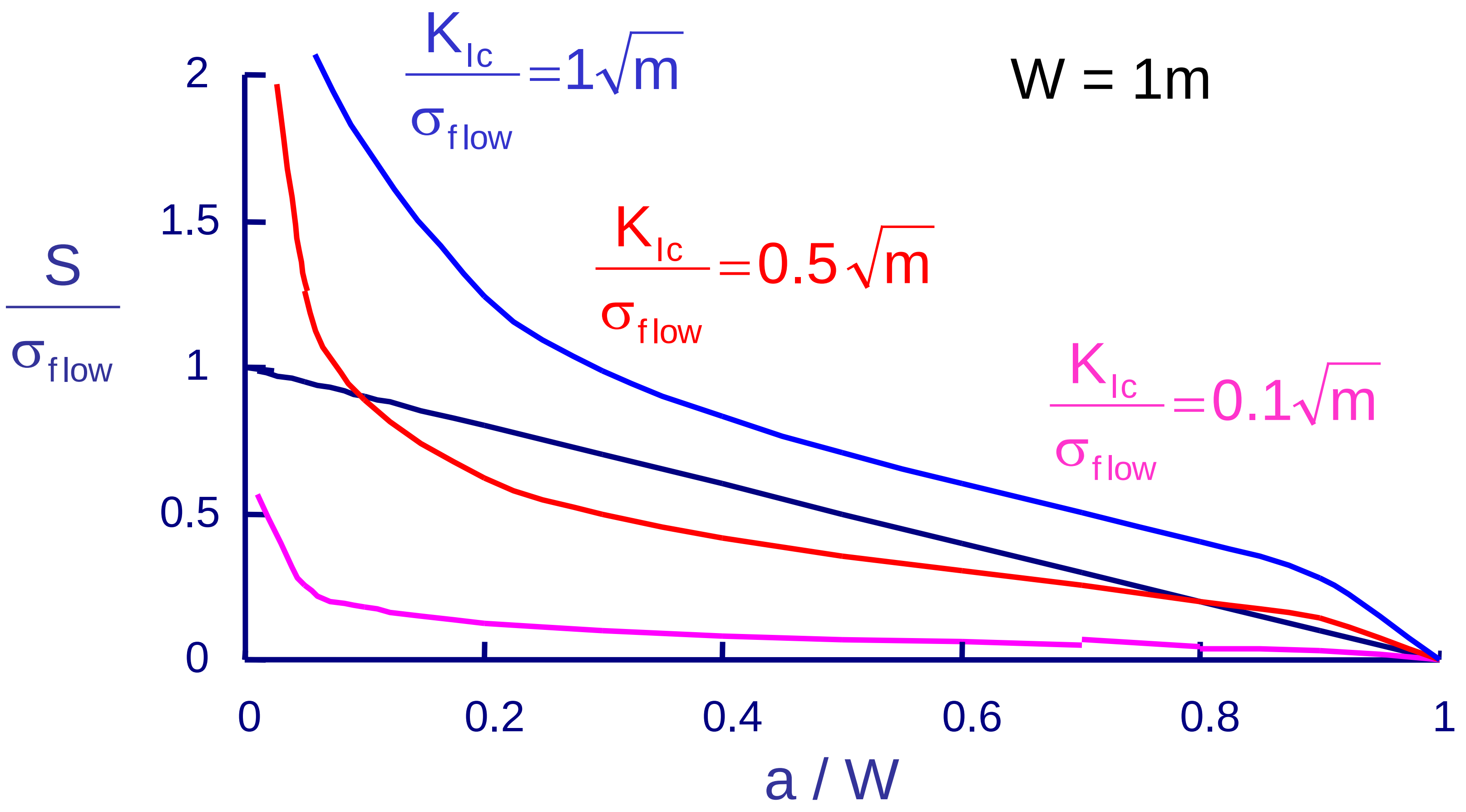
$$\sigma_{\text{flow}}\left(1-\frac{a}{W}\right)=\frac{K_{Ic}}{\sqrt{\pi\frac{a}{w}}\sqrt{W}\,f\left(\frac{a}{W}\right)}$$

$$\frac{K_{Ic}}{\sigma_{\text{flow}}}=\left(1-\frac{a}{W}\right)\sqrt{\pi\frac{a}{w}}\sqrt{W}\,f\left(\frac{a}{W}\right)$$

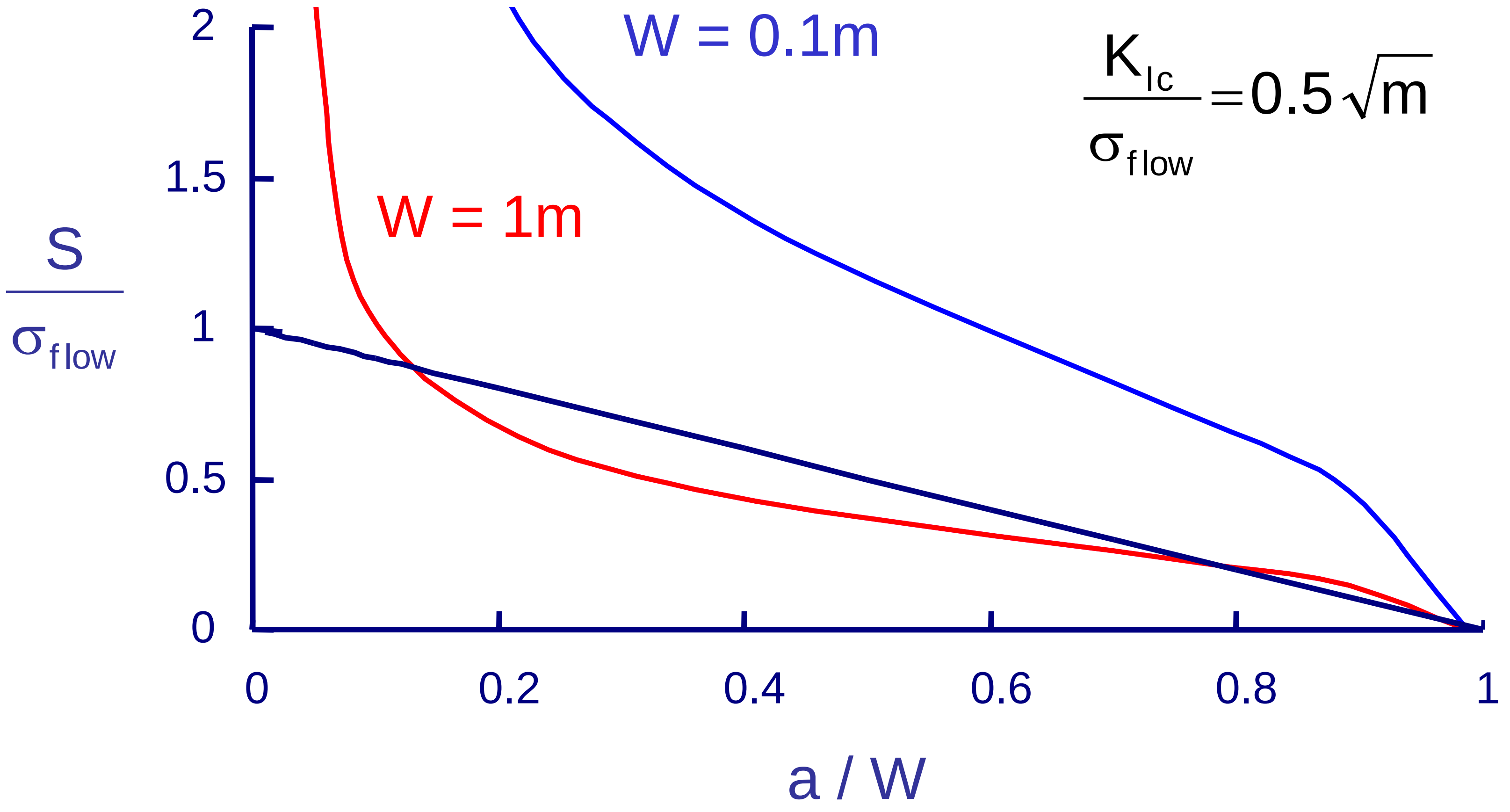
Material Properties

	σ_{ys}	K_{Ic}	K_{Ic}/σ_{ys}
1020	250	200	0.800
2024-T3	345	44	0.128
7075-T6	495	25	0.051
Ti-6Al-4V	910	105	0.115
Ti-6Al-4V	1035	55	0.053
4340	860	99	0.115
4340	1510	60	0.040
17-7 PH	1435	77	0.054
52100	2070	14	0.007

Fracture Diagram



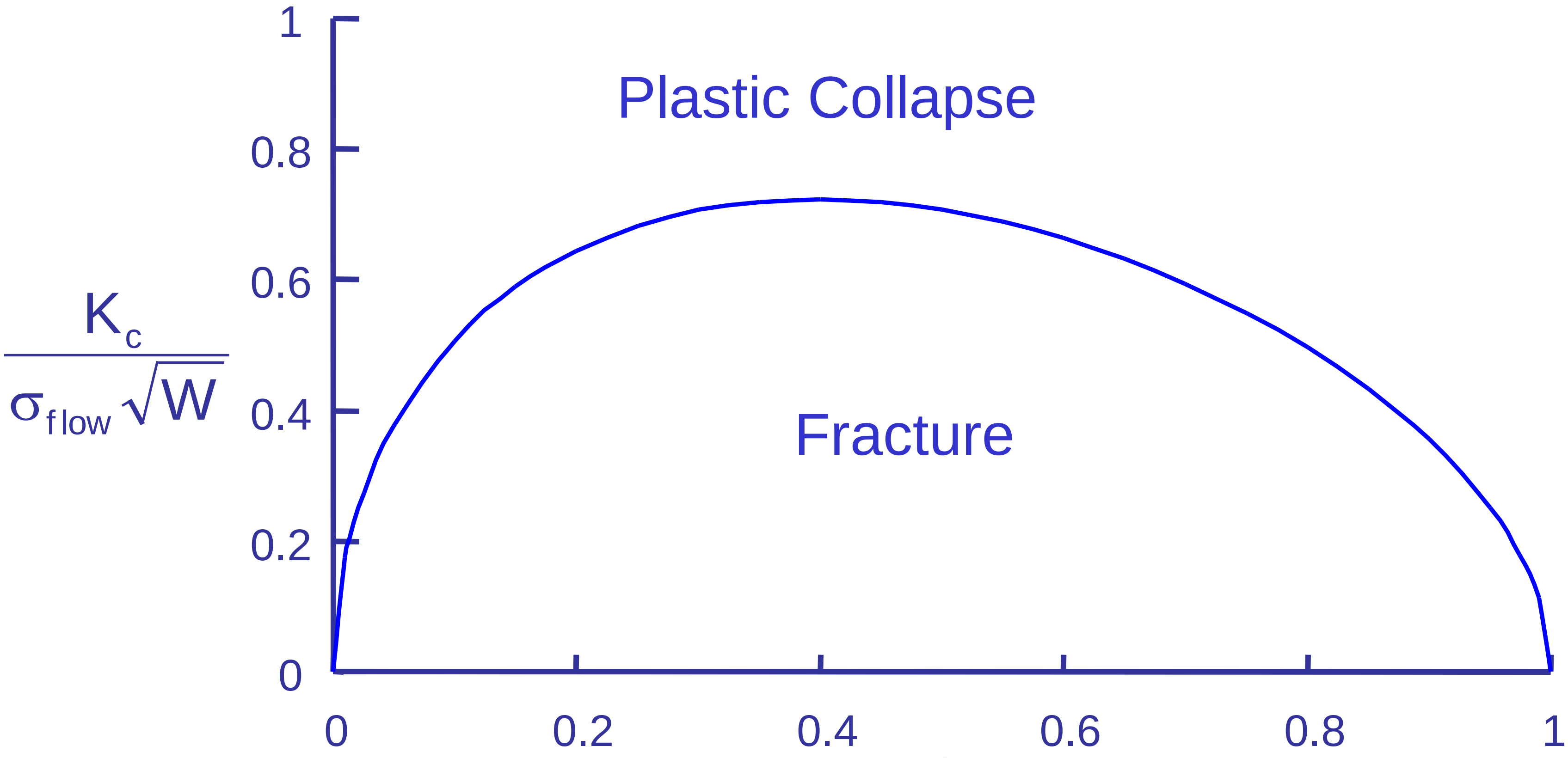
Fracture Diagram



Fracture vs. Collapse

$$\frac{K_{Ic}}{\sigma_{flow}\sqrt{W}} = \left(1 - \frac{a}{W}\right)\sqrt{\pi\frac{a}{W}}\sqrt{\frac{2W}{\pi a}\tan\frac{\pi a}{2W}}$$

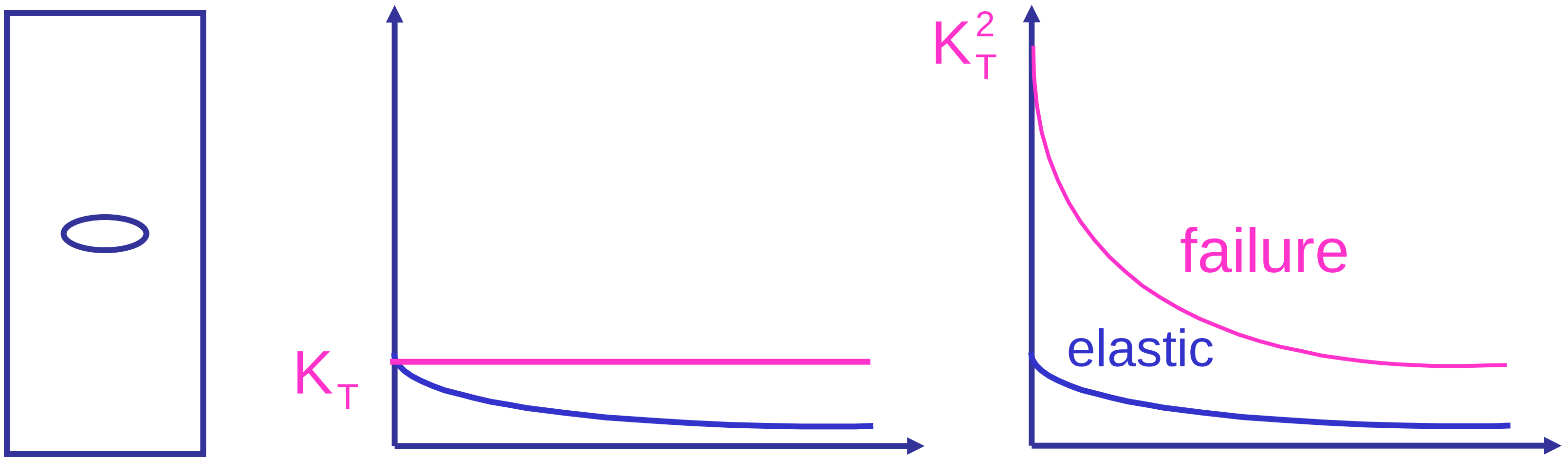
Fracture Diagram



small flaws
high stresses

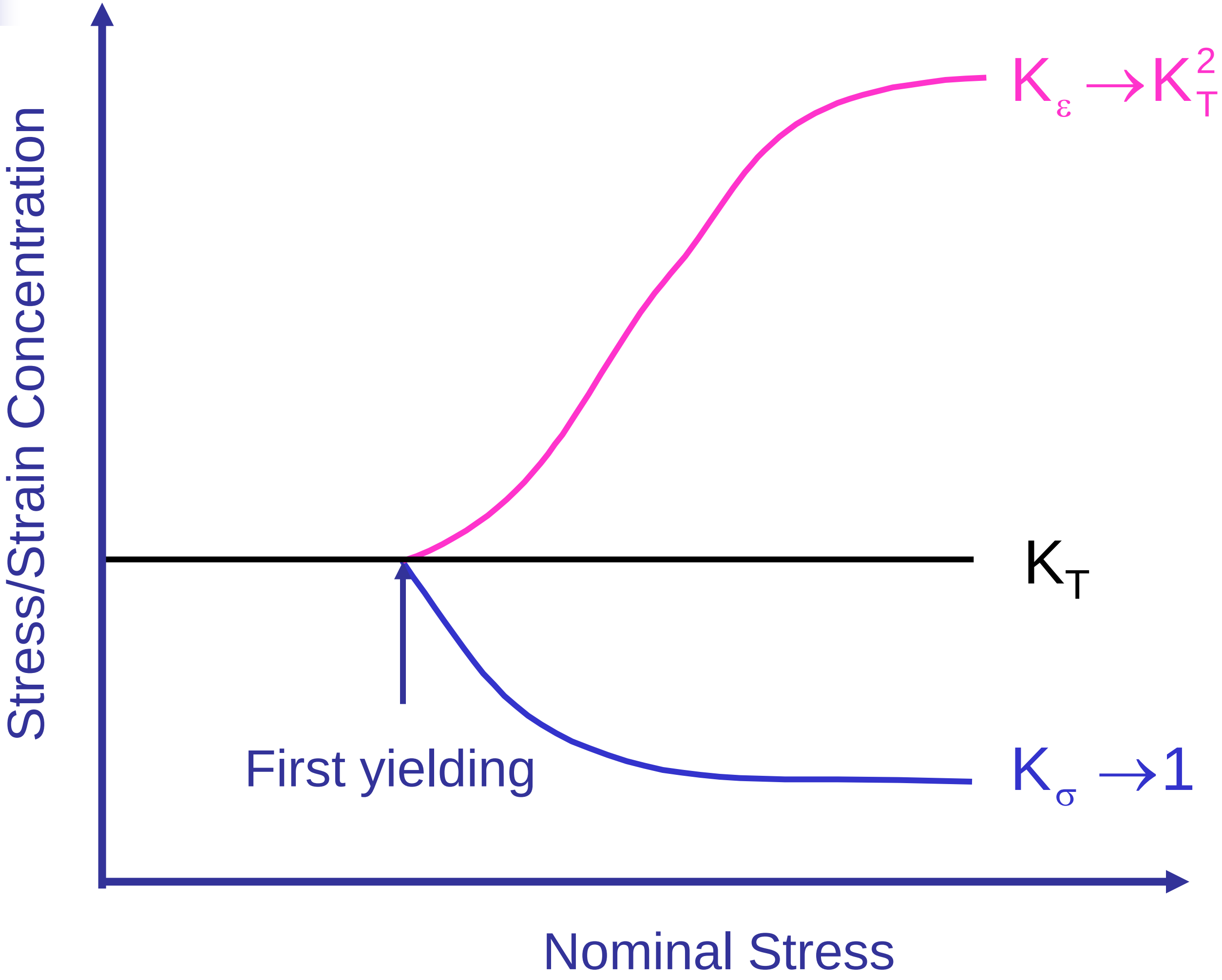
large flaws
low stresses

Typical Stress Concentration



Stress distribution Strain distribution

Stress and Strain Concentration



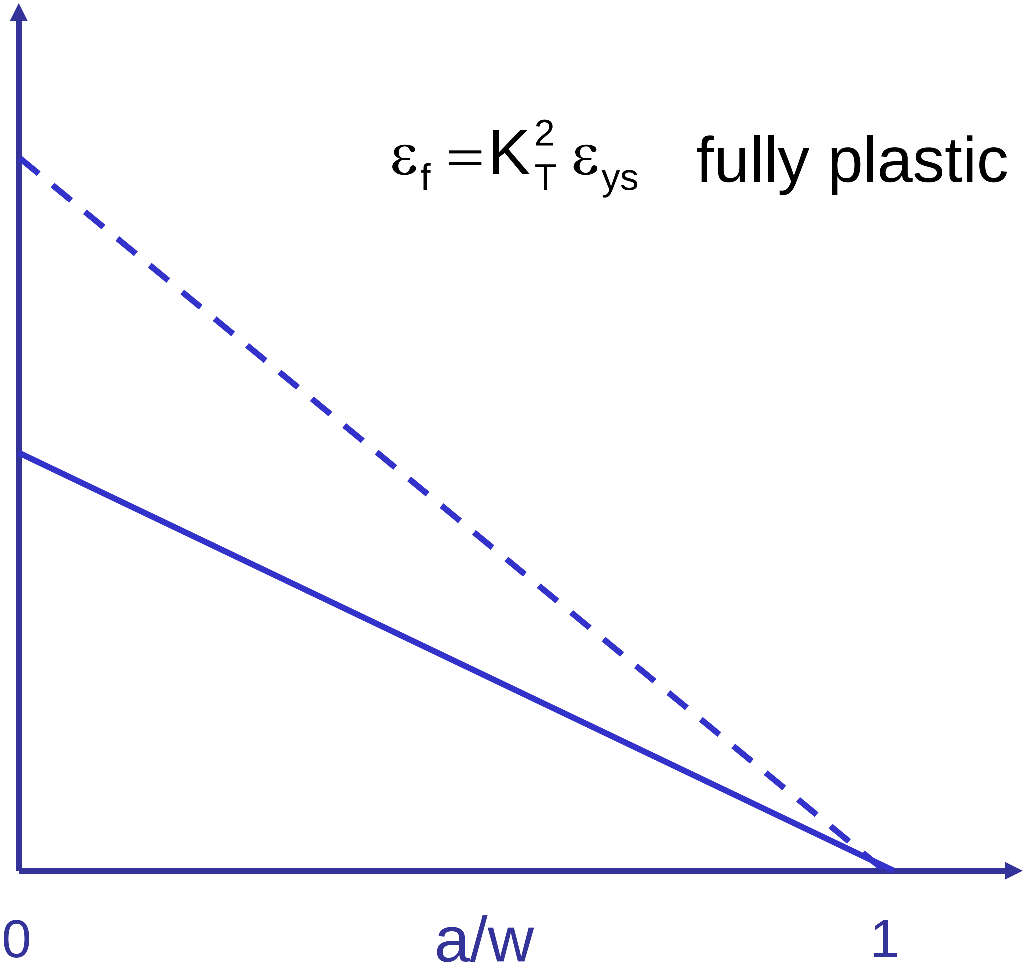
Failure Diagram – Ductile Material

ε_f is large

$$P_{max} = \frac{A_{net}\ \varepsilon_f\, E}{K_T^2}$$

strength limited

$$P_{max} = \sigma_{flow}\ A_{net}$$



Strength limit is reached before cracking at the notch

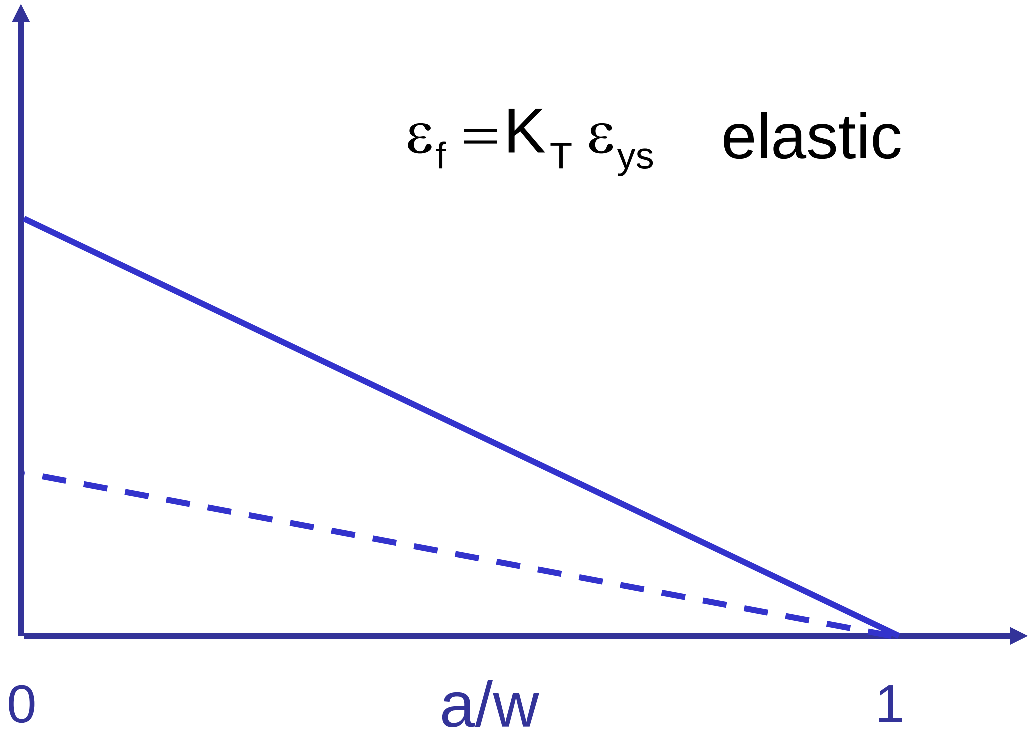
Failure Diagram – Brittle Material

$$P_{max} = \sigma_{flow} A_{net}$$

ε_f is small

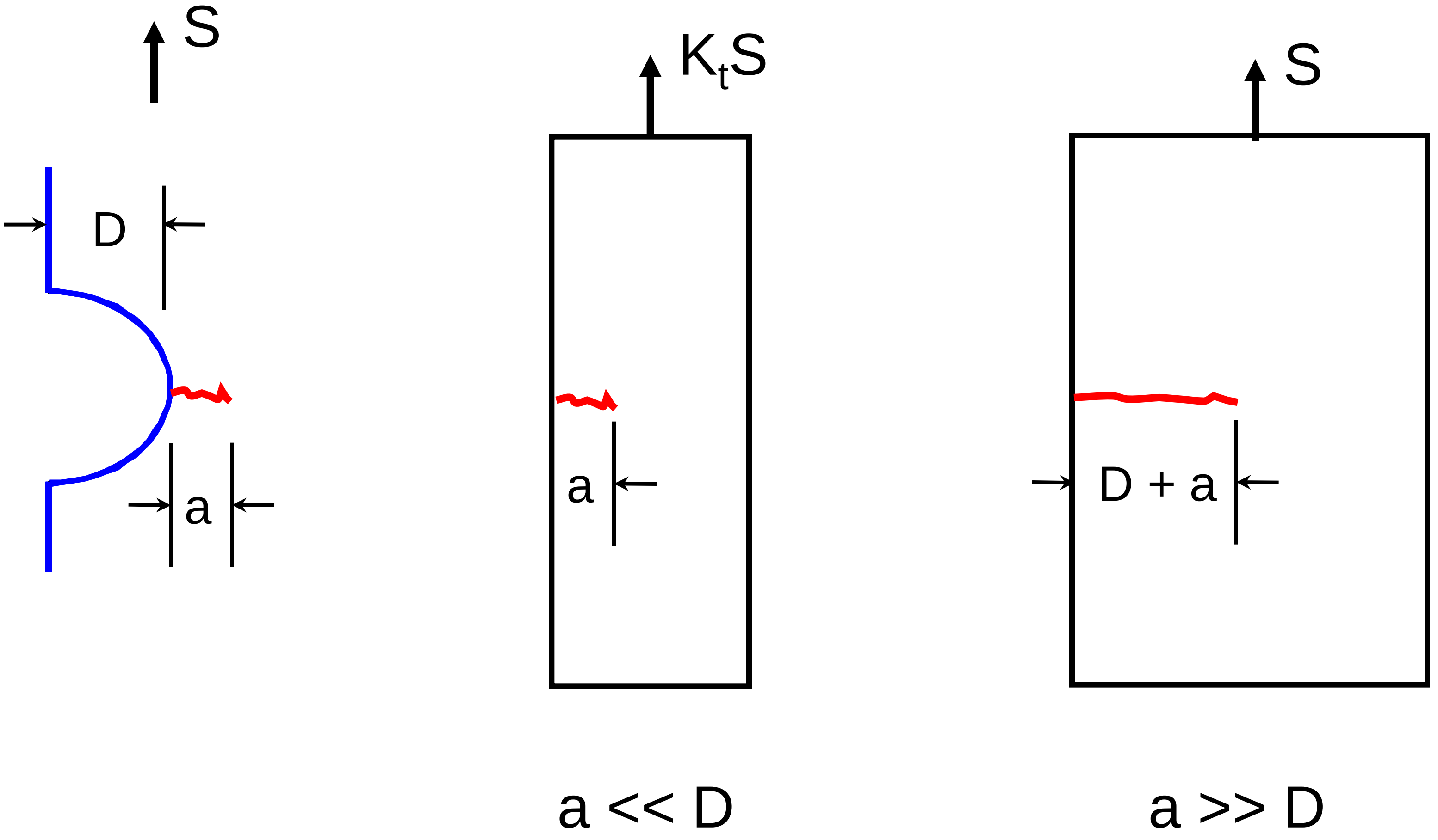
$$P_{max} = \frac{A_{net} \varepsilon_f E}{K_T}$$

ductility limited

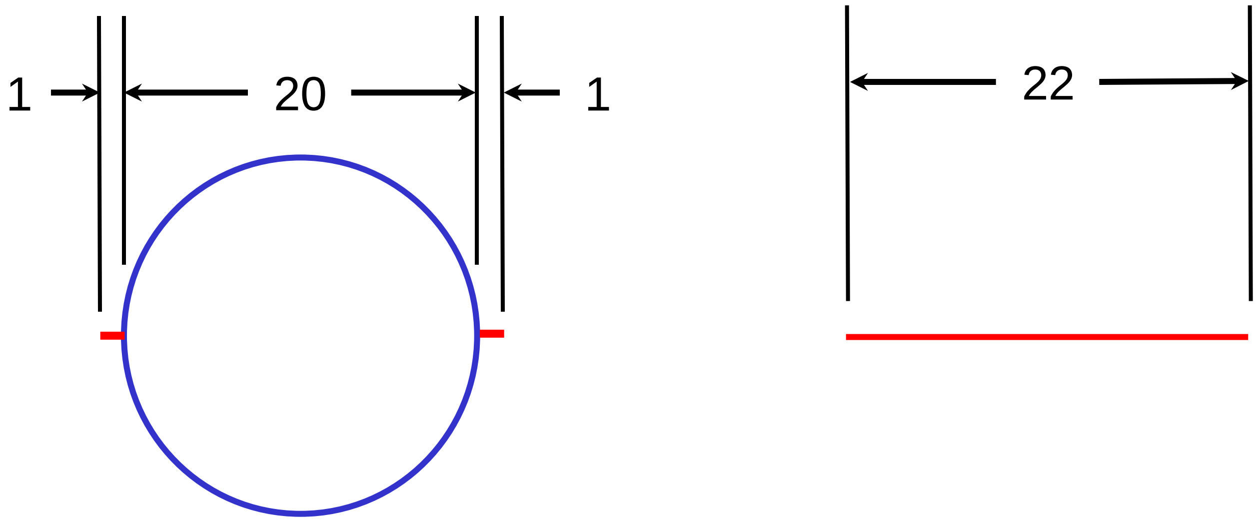


Cracks form at the notch before the limit load is reached

Cracks at Notches



Cracks at Holes



Once a crack reaches 10% of the hole radius, it behaves as if the hole was part of the crack

Notch Fracture

What ratio of strength to toughness is needed to avoid fracture?

$$K_c = \sigma_{ys} 1.12\, K_T \sqrt{\pi 0.1r}$$

For $K_T = 3$ and $r = 10$ mm

$$\frac{K_c}{\sigma_{ys}} > 0.18$$

to avoid fracture from the notch

Material Properties

	σ_{ys}	K_{Ic}	K_{Ic}/σ_{ys}
1020	250	200	0.800
2024-T3	345	44	0.128
7075-T6	495	25	0.051
Ti-6Al-4V	910	105	0.115
Ti-6Al-4V	1035	55	0.053
4340	860	99	0.115
4340	1510	60	0.040
17-7 PH	1435	77	0.054
52100	2070	14	0.007

Summary

- Fracture is a likely failure mode for all higher strength materials
- Fracture is even more likely at stress concentrators

Fatigue and Fracture